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Regularized theta liftings and periods of modular functions

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Abstract. In this paper, we use regularized theta liftings to construct weak Maass forms of weight 1/2 as lifts of weak Maass forms of weight 0. As a special case we give a new proof of some of recent results of Duke, Toth and the third author on cycle integrals of the modular j-invariant and extend these to any congruence subgroup. Moreover, our methods allow us to settle the open question of a geometric interpretation for periods of j along infinite geodesics in the upper half plane. In particular, we give the 'central value' of the (non-existent) 'L-function' for j. The key to the proofs is the construction of a kind of simultaneous Green function for both the CM points and the geodesic cycles, which is of independent interest.

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1. Introduction

In this paper we use the theta correspondence to study the traces and periods of modular functions. To place our work in context, we begin by reviewing recent results of Zagier [30,31], of Duke, Toth and the third author [9], and of [7].

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Generating series of traces of singular moduli. For a non-zero integer d, let $\mathcal{Q}_d$ be the set of integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$, where we take Q to be positive definite for $d < 0$. We write $Q = [a, b, c]$. The natural action of $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ divides $\mathcal{Q}_d$ into finitely many classes. For $d < 0$ and $Q \in \mathcal{Q}_d$, the root

$$z_Q = \frac{-b + \sqrt{d}}{2a}$$

defines a CM point in the upper half plane $\mathbb{H}$. The values of the classical j-invariant at the CM points have been of classical interest. For $f \in M_0^! = \mathbb{C}[j]$, the space of weakly holomorphic functions of weight 0 for $\Gamma(1)$, we define for $d < 0$, the modular trace of f by

$$\mathrm{tr}_d(f) = \sum_{Q \in \Gamma(1)\backslash \mathcal{Q}_d} \frac{1}{|\Gamma(1)_Q|} f(z_Q). \tag{1.1}$$

Here $\Gamma(1)_Q$ denotes the finite stabilizer of Q. The theory of such traces has enjoyed renewed interest thanks to the work of Borcherds [4] and Zagier [31], where connections between modular traces, automorphic infinite products and weakly holomorphic modular forms of half-integral weight are established. In particular, a beautiful theorem of Zagier [31] shows for $j_1 := j - 744$ that the generating series

$$g_1(\tau) := -q^{-1} + 2 + \sum_{d<0} \mathrm{tr}_d(j_1) q^{-d} = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + \cdots \tag{1.2}$$

is a weakly holomorphic modular form of weight $3/2$ for the Hecke subgroup $\Gamma_0(4)$. Here $q = e^{2\pi i \tau}$ with $\tau = u + iv \in \mathbb{H}$.

On the other hand, an older result of Zagier [30] on the Hurwitz–Kronecker class numbers $H(|d|) = \mathrm{tr}_d(1)$ states that

$$g_0(\tau) := -\frac{1}{12} + \sum_{d<0} \mathrm{tr}_d(1) q^{-d} + \frac{1}{16\pi\sqrt{v}} \sum_{n=-\infty}^{\infty} \beta_{3/2}(4\pi n^2 v) q^{-n^2} \tag{1.3}$$

is a harmonic weak Maass form of weight $3/2$ for $\Gamma_0(4)$. Here

$$\beta_k(s) = \int_1^\infty e^{-st} t^{-k}\, dt.$$

Using the methods of [13], in [7] the first and the second author unified and generalized (1.2) and (1.3) to traces of arbitrary weakly holomorphic modular functions f of weight zero on modular curves $\Gamma\backslash\mathbb{H}$ for any congruence subgroup Γ. The results in [7] are obtained by considering a theta lift

$$I_{3/2}(\tau, f) = \int_{\Gamma\backslash\mathbb{H}} f(z) \cdot \Theta_L(\tau, z, \varphi_{KM})\, d\mu(z) \tag{1.4}$$

of f against a theta series associated to an even lattice L of signature $(1, 2)$ and the Kudla–Millson Schwartz function φ_{KM} of weight $3/2$, see [20]. Here $d\mu(z) = \frac{dx\,dy}{y^2}$. The integral converges, since the decay of the theta kernel turns out to be faster than the exponential growth of f. For $f = j_1$ and $f = 1$ and the appropriate choice of the lattice L one obtains the generating series (1.2) and (1.3) above, while in addition giving a geometric interpretation to the Fourier coefficients of non-positive index.

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Cycle integrals of modular functions. In a different direction, turning to the natural question of the case of positive discriminants, Duke, Toth and the third author [9] recently studied the cycle integrals of modular functions as analogs of singular moduli. Their work gives an extension and generalization of the results of Borcherds and Zagier. For $d > 0$, the two roots of $Q \in \mathcal{Q}_d$ lie in $\mathbb{P}^1(\mathbb{R})$, and we let c_Q be the properly oriented geodesic in $\mathbb{H}$ connecting these roots. For non-square $d > 0$, the stabilizer $\Gamma(1)_Q$ is infinitely cyclic, and we set $C_Q = \Gamma(1)_Q \backslash c_Q$. Then C_Q defines a closed geodesic on the modular curve. For $f \in M_0^!$ and in analogy with (1.1) let

$$\operatorname{tr}_d(f) = \frac{1}{2\pi} \sum_{Q \in \Gamma(1) \backslash \mathcal{Q}_d} \int_{C_Q} f(z) \frac{dz}{Q(z,1)}. \tag{1.5}$$

One of the main results of [9] realizes the generating series of traces of both the CM values and the cycle integrals for any $f \in M_0^!$ as a form of weight $1/2$. More precisely, for $f = j_1$ the generating series

$$\begin{aligned} h_1(\tau) := \sum_{d>0} \operatorname{tr}_d(j_1)\, q^d &+ 2\sqrt{v}\beta^c_{\frac{1}{2}}(-4\pi v)q - 8\sqrt{v} \\ &+ 2\sqrt{v} \sum_{d<0} \operatorname{tr}_d(j_1)\beta_{\frac{1}{2}}(4\pi|d|v)q^d \end{aligned} \tag{1.6}$$

is a harmonic weak Maass form of weight 1/2 for $\Gamma_0(4)$. Here

$$\beta^c_{1/2}(s) = \int_0^1 e^{-st} t^{-1/2}\, dt$$

is the 'complementary' function to $\beta_{1/2}(s)$. The analog of (1.3) for $f = 1$ is that

$$\begin{aligned} h_0(\tau) := \sum_{d>0} \operatorname{tr}_d(1)\, q^d &+ \frac{\sqrt{v}}{3} + 2\sqrt{v} \sum_{d<0} \operatorname{tr}_d(1)\beta_{\frac{1}{2}}(4\pi|d|v)q^d \\ &+ \sum_{n \neq 0} \alpha(4n^2 v) q^{n^2} - \frac{1}{4\pi}\log v \end{aligned} \tag{1.7}$$

is a weak Maass form of weight 1/2 for $\Gamma_0(4)$. Here

$$\alpha(s) = \frac{\sqrt{s}}{4\pi} \int_0^\infty \log(1+t) e^{-\pi s t} t^{-1/2}\, dt.$$

Duke, Toth, and the third author prove their results by first constructing an explicit basis for the space of harmonic weak Maass forms of weight 1/2 constructed out of Poincaré series. The construction of such a basis is quite delicate, due in part to the residual spectrum. Then (1.6) is proved by explicitly computing the cycle integrals of weight 0 non-holomorphic Poincaré series in terms of exponential sums and then relating these to Kloosterman sums. The construction of $h_0(z)$ is similar using a Kronecker limit type formula for the weight $1/2$ Eisenstein series.

The functions obtained in [30] and [31] and the ones in [9] are related via the differential operator

$$\xi_{1/2} = 2iv^{1/2}\overline{\frac{\partial}{\partial \bar{\tau}}},$$

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which maps forms of weight $1/2$ to the dual weight $3/2$. We have

$$\xi_{1/2}(h_1) = -2g_1 \quad \text{and} \quad \xi_{1/2}(h_0) = -2g_0. \tag{1.8}$$

In this way, the results of Duke, Toth, and the third author contain (for $\mathrm{SL}_2(\mathbb{Z})$) the previous work on modular traces.

We note that in contrast to (1.2) and (1.3), in the formulas (1.6) and (1.7), the sums over CM points now occur in the Fourier coefficients for negative d, while cycle integrals occur in the Fourier coefficients for positive d. The ξ operator annihilates the holomorphic cycle integral terms and shifts the CM traces to the coefficients with positive index.

The coefficients of square index. For *square* discriminants d, no definition for the modular trace tr_d in (1.5) is given in [9], and hence the geometric meaning of the corresponding coefficients $\mathrm{tr}_d(j_1)$ and $\mathrm{tr}_d(1)$ in (1.6) and (1.7) is left open. Rather, for d a square, these terms represent in [9] only the unknown d-th Fourier coefficient of the residual Poincaré series which define the generating series. Analytically, the Fourier coefficients of square index d are exactly where the weight $1/2$ Poincaré series have poles, and residual terms have to be subtracted to obtain h_1 and h_0 (see [9, Lemma 3, (2.26) and (2.27)]). This makes them rather intractable to compute. Geometrically, the issue is that the stabilizer $\Gamma(1)_Q$ is trivial for square d and hence the cycle C_Q corresponds to an infinite geodesic in the modular curve. Therefore the integral of a modular function over C_Q does not converge. This represents the principle obstacle to a geometric definition of the trace analogous to (1.5). In fact, the authors of [9] raise the questions whether their results can be approached using theta lifts as in [7] and whether one can give more insight to the mysterious nature of the square coefficients.

In this paper, we indeed use the theta machinery to study the traces and periods of modular functions. In particular, we succeed in computing the coefficients of square index in (1.6) and (1.7) and to give a geometric interpretation for those terms. In fact, we consider the modular traces and periods for any (harmonic) weak Maass form on any modular curve $\Gamma\backslash\mathbb{H}$ defined by a congruence subgroup Γ.

The central L-value of the j-invariant. We first explain how to define the modular trace for $\Gamma(1)$ for square index d. In view of (1.5) it suffices to regularize the period

$$\int_{C_Q} f(z)\frac{dz}{Q(z,1)}$$

whenever C_Q is an infinite geodesic. We will do this here only when $Q = [0, \sqrt{d}, 0]$ such that $Q(z,1) = \sqrt{d}z$ and C_Q is (the image of) the imaginary axis. In fact, for $d = 1$, we have $C_1 = C_Q$. Hence the problem reduces to defining the 'central value' of the (non-existent) L-function for f. Note that if there were a cusp form

$$f(z) = \sum_{n>0} a(n)e^{2\pi i n z}$$

of weight 0, then the cycle integral of f over C_Q would converge and equal the value of its L-function at $s = 0$ given by

$$\int_{C_Q} f(z)\frac{dz}{z} = 2\int_1^\infty f(iy)\frac{dy}{y} = 2\sum_{n\neq 0} a(n)\int_{2\pi n}^\infty e^{-t}\frac{dt}{t}. \tag{1.9}$$

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In analogy to this, for $f \in M_0^!$, we define

$$\int_{C_Q}^{\mathrm{reg}} f(z)\frac{dz}{z} := 2\sum_{n\neq 0} a(n)\mathcal{E}\mathcal{I}(2\pi n), \tag{1.10}$$

where $\mathcal{E}\mathcal{I}(w)$ is related to the exponential integrals defined in [1, Sections 5.1.1 and 5.1.2] by

$$\mathcal{E}\mathcal{I}(w) := \int_w^\infty e^{-t}\frac{dt}{t} = \begin{cases} E_1(w) & \text{if } w > 0, \\ -\operatorname{Ei}(-w) & \text{if } w < 0. \end{cases} \tag{1.11}$$

Here in the second case the integral is defined as the Cauchy principal value.

A more geometric characterization for the regularized period is given by the following result.

Theorem 1.1. *Let $f \in M_0^!$ be a modular function. Then for C_Q the imaginary axis as above and, for any $T > 0$,*

$$\int_{C_Q}^{\mathrm{reg}} f(z)\frac{dz}{z} = 2\int_i^{iT} f(z)\frac{dz}{z} - \int_{iT}^{iT+1} f(z)(\psi(z)+\psi(1-z))\,dz.$$

Here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the Euler Digamma function.

This formula is strikingly similar to the one in [14, Lemma 4.3 and Theorem 4.4], where the critical L-values of a modular form (not necessarily cuspidal) are interpreted as cohomological periods of holomorphic 1-forms with values in a local system over certain closed 'spectacle' cycles. We will discuss the cohomological interpretation of the cycle integrals of this paper elsewhere.

For $T = 1$ and using that j_1 has real Fourier coefficients we obtain the following beautiful formula for the regularized integral of j_1 along the imaginary axis. We have

$$\int_{C_Q}^{\mathrm{reg}} j_1(z)\frac{dz}{z} = -2\operatorname{Re}\left(\int_i^{i+1} j_1(z)\psi(z)\,dz\right). \tag{1.12}$$

In fact, this formula was first suggested to us by D. Zagier, who obtained (1.12) based on heuristic arguments and verified numerically that defining $\operatorname{tr}_1(j_1)$ using (1.12) gives the correct value for $\operatorname{tr}_1(j_1)$ in (1.6).

The main result. Before we describe the theta lift we employ, we first state our main result in a special case. Let p be a prime (or $p = 1$). We consider the set $\mathcal{Q}_{d,p}$ of quadratic forms $[a,b,c] \in \mathcal{Q}_d$ such that $a \equiv 0 \pmod{p}$. The group $\Gamma_0^*(p)$, the extension of the Hecke group $\Gamma_0(p) \subset \Gamma(1)$ with the Fricke involution $W_p = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$, acts on $\mathcal{Q}_{d,p}$ with finitely many orbits. Let $f \in M_0^!(\Gamma_0^*(p))$ be a weakly holomorphic modular function of weight 0 for $\Gamma_0^*(p)$. We define the modular trace of f of index $d \neq 0$ by

$$\operatorname{tr}_d(f) = \begin{cases} \displaystyle\sum_{Q\in\Gamma_0^*(p)\backslash\mathcal{Q}_{d,p}} \frac{1}{|\Gamma_0^*(p)_Q|} f(\alpha_Q) & \text{if } d < 0, \\ \displaystyle\frac{1}{2\pi}\sum_{Q\in\Gamma_0^*(p)\backslash\mathcal{Q}_{d,p}} \int_{\Gamma_0^*(p)_Q\backslash c_Q}^{\mathrm{reg}} f(z)\frac{dz}{Q(z,1)} & \text{if } d > 0. \end{cases} \tag{1.13}$$

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Theorem 1.2. *Let $f(z) = \sum_{n \gg -\infty} a(n)e(nz) \in M_0^!(\Gamma_0^*(p))$ with $a(0) = 0$. Then*

$$H(\tau, f) := \sum_{d>0} \mathrm{tr}_d(f)\, q^d + 2\sqrt{v} \sum_{m>0} \sum_{n<0} a(mn)\beta^c_{\frac{1}{2}}(-4\pi m^2 v) q^{m^2} - 2\sqrt{v}\, \mathrm{tr}_0(f)$$
$$+ 2\sqrt{v} \sum_{d<0} \mathrm{tr}_d(f) \beta_{\frac{1}{2}}(4\pi |d| v) q^d$$

is a harmonic weak Maass form of weight $1/2$ for $\Gamma_0(4p)$. Here

$$\mathrm{tr}_0(f) = -\frac{1}{2\pi} \int^{\mathrm{reg}}_{\Gamma_0^*(p)\backslash \mathbb{H}} f(z)\, d\mu(z) = 4 \sum_{n>0} a(-n)\sigma_1(n).$$

is the suitably regularized average value of f on $\Gamma_0^*(p)\backslash\mathbb{H}$, defined in (3.12).

For $p = 1$ and $f = j_1$ we recover $h_1(\tau)$, now with explicit geometric formulas for the square coefficients in the generating series. For $f = 1$ we have a similar theorem which generalizes h_0. The statements for any congruence subgroup are formulated in terms of quadratic spaces of signature (2,1).

The regularized theta lift. To prove our results we study the theta integral

$$I_{1/2}(\tau, f) = \int_{\Gamma\backslash\mathbb{H}} f(z) \cdot \Theta_L(\tau, z, \varphi_0)\, d\mu(z). \tag{1.14}$$

Here the kernel function $\Theta_L(\tau, z, \varphi_0)$ is the Siegel theta series associated to the standard Gaussian φ_0 of weight $1/2$ for a rational quadratic space of signature $(2, 1)$. It is related to the kernel of (1.4) via

$$\xi_{1/2}(\Theta_L(\tau, z, \varphi_0)) = -\frac{1}{\pi}\Theta_L(\tau, z, \varphi_{KM}), \tag{1.15}$$

and we obtain formally the same relation for the theta lifts $I_{1/2}$ and $I_{3/2}$, which matches (up to a constant) the relations given in (1.8). When f is a Maass cusp form, the integral (1.14) converges, and this lift has been previously studied by Maass [23], Duke [8], and Katok and Sarnak [17] among others. However, the theta kernel $\Theta_L(\tau, z, \varphi_0)$ in contrast to the one used for $I_{3/2}$ in [7] is now moderately *increasing*. Hence when the input function f is not a cusp form, the integral does not converge (even for $f = 1$) and has to be regularized.

We analyze in detail two different approaches to regularize (1.14) for any weak Maass form $f(z)$ of weight 0 for Γ with eigenvalue λ under the Laplace operator Δ_z. The case $\lambda = 0$ is the most interesting, which we now describe. First, following an idea of Borcherds [4] and Harvey–Moore [15], we regularize the integral by integrating over a truncated fundamental domain $\mathcal{F}_T$ for $\Gamma\backslash\mathbb{H}$ and taking a limit. More precisely, for a complex variable s we consider

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} f(z) \cdot \Theta_L(\tau, z, \varphi_0)\, y^{-s}\, d\mu(z), \tag{1.16}$$

where $\mathcal{F}_T$ is a suitable truncated fundamental domain for Γ. For the real part of s sufficiently large, the limit converges and the resulting holomorphic function of s defined in a right half-plane admits a meromorphic continuation to the whole s-plane. Then we regularize (1.14) by taking the constant term in the Laurent expansion of (1.16) at $s = 0$.

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The second approach uses differential operators in the spirit of the regularized Siegel–Weil formula of Kudla and Rallis [22]. Using Eisenstein and Poincaré series of weight 0 one can construct a 'spectral deformation' of f, that is, a family of functions $f_s(z)$ such that $\Delta_z f_s = s(1-s) f$ and $f_1 = f$, and then we consider

$$\frac{1}{s(1-s)} \int_{\Gamma\backslash\mathbb{H}} f_s(z) \cdot \Delta_z \Theta_L(\tau, z, \varphi_0)\, d\mu(z). \tag{1.17}$$

The point is that $\Delta_z \Theta_L(\tau, z, \varphi_0)$ is of very rapid decay (like the kernel for (1.4)) and hence the integral converges. Furthermore, by the adjointness of the Laplace operator we see that (1.17) formally equals (1.14). Then we can regularize (1.14) to be the constant term in the Laurent expansion of (1.17) at $s = 1$. The relation between the two regularizations can be described as follows.

Theorem 1.3. *Let f be a harmonic weak Maass form of weight* 0 *for a congruence subgroup Γ. If the constant terms of the Fourier expansion of f at all cusps of Γ vanish, then the two regularizations of* (1.14) *coincide. Otherwise, they differ by an explicit linear combination of holomorphic unary Jacobi theta series of weight* $1/2$.

By lifting Poincaré series of weight 0, we are also able to realize the Poincaré series of weight $1/2$ which occur in [9] as theta lifts, explicitly relating our approach to the one of Duke, Toth, and the third author.

The Green function η and the Fourier expansion of the theta lift. The key to computing the Fourier coefficients of the lift $I_{1/2}(\tau, f)$ is the construction of a Green function η for the Schwartz function φ_0. We view the set of all rational quadratic forms $Q = [a, b, c]$ together with the discriminant form $d = b^2 - 4ac$ as a quadratic space $\mathcal{Q}$ of signature $(2, 1)$ whose associated symmetric space is equivalent to $\mathbb{H}$. In this way, φ_0 can be regarded as a function on $\mathcal{Q} \times \mathbb{H}$. We explicitly construct a singular function $\eta(Q, z)$ for all Q of non-zero discriminant such that

$$\Delta_z \eta(Q, z) = -\frac{1}{4\pi} \varphi_0(Q, z)$$

outside the singularities of η. We call η a Green function for φ_0. If $d < 0$, then $\eta(Q)$ has a logarithmic singularity at the point z_Q, while for $d > 0$, the function $\eta(Q)$ is differentiable, but not C^1, and the discontinuity of $\partial\eta$ exactly occurs at the geodesic cycle c_Q.

Theorem 1.4. *Let Q be a integral binary quadratic form with discriminant $d \neq 0$, not a square, with stabilizer Γ_Q in Γ. Then for f a weak Maass form of weight* 0 *with eigenvalue λ, the integral $\int_{\Gamma_Q\backslash\mathbb{H}} f(z)\varphi_0(Q, z)\, d\mu(z)$ converges, and*

$$\int_{\Gamma_Q\backslash\mathbb{H}} f(z)\varphi_0(Q, z)\, d\mu(z) = -\frac{1}{4\pi}\lambda \int_{\Gamma_Q\backslash\mathbb{H}} f(z)\eta(Q, z)\, d\mu(z) + \begin{cases} \left(\frac{1}{|\Gamma_Q|} f(z_Q)\right) 2\beta_{\frac{1}{2}}(4\pi|d|) e^{-2\pi d} & \text{if } d < 0, \\ \left(\int_{c_Q} f(z) \frac{dz}{Q(z,1)}\right) e^{-2\pi d} & \text{if } d > 0. \end{cases}$$

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This essentially computes the Fourier coefficients of non-square index for $I_{1/2}(\tau, f)$ (at least when $\lambda = 0$). For the other Fourier coefficients we also utilize η. The square coefficients however require some rather intricate considerations since in that case Γ_Q is trivial and $\int_{\Gamma_Q \backslash \mathbb{H}} f(z)\varphi_0(Q, z)\, d\mu(z)$ does not converge.

The existence of such a Green function is rather surprising, and we believe is of independent interest. Moreover, η refines Kudla's Green function ξ for φ_{KM} (see [18]) which played a crucial role in studying (1.4) in [7]. However, ξ only has singularities along the CM points and hence cannot detect the periods over the geodesic cycles. Note that ξ plays an important role in the Kudla program (see e.g. [19]) which is concerned with the realization of generating series in arithmetic geometry as automorphic forms, in particular as the derivative of Eisenstein series. It is therefore an interesting question how the results of this paper and η in particular fit into this framework.

Other input. One can also study other input functions f for the lift $I_{1/2}(\tau, f)$. One natural extension is to consider a meromorphic function f of weight 0 with at most simple poles in $\mathbb{H}$. For example, for $d > 0$ a non-square, taking the d-th coefficient (in τ) of the lift of $f_w(z) = j'(w)/(j(z) - j(w))$ (with $w \in \mathbb{H}$), one obtains a non-holomorphic form of weight 2 (in w). Using the techniques of this paper one can prove that this form is a 'completion' of the holomorphic generating series

$$F_d(w) = -\sum_{m \geq 0} \operatorname{tr}_d(j_m) e^{2\pi i m w},$$

which in [9, Theorem 5] is shown to be a holomorphic modular integral of weight 2 with a rational period function. Here $\{j_m\}$ denotes the unique basis of $M_0^!$ whose members are of the form

$$j_m(z) = e^{-2\pi i m z} + O(e^{2\pi i z}).$$

The higher weight situation has been considered recently in [2, 3].

Another interesting case is when $f = \log \|F\|$ is the logarithm of the Petersson metric of a meromorphic modular form F. For example, for $F(z) = \Delta(z)$, the discriminant function, one can show (similarly as in [7, Theorem 1.2]) that $I_{1/2}(\tau, f)$ is equal to the constant term in the Laurent expansion at $s = 1$ of the derivative of an Eisenstein series of weight $1/2$. In general, one can view the lift of such input as the adjoint of the (additive) Borcherds lift, which uses the same kernel function $\Theta_L(\tau, z, \varphi_0)$. We will consider these lifts in a different paper.

The theta lift has been studied recently also by Matthes [24] using the second regularization via differential operators. More precisely, he considers the analogous lift for general hyperbolic n-space. In our case, he considers (mainly) input functions with non-zero eigenvalue under the Laplace operator and employs a different method to compute the coefficients of non-square index leaving the square coefficients open.

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2. Vector-valued modular forms for SL_2

Let N be a positive integer. Let (V, Q) be the 3-dimensional quadratic space over $\mathbb{Q}$ given by the trace zero 2×2 matrices

$$V := \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Q}) \right\}, \tag{2.1}$$

with the quadratic form $Q(X) = -N \det(X)$. The corresponding bilinear form is

$$(X, Y) = N \operatorname{tr}(XY),$$

and its signature is $(2, 1)$. We let $G = \mathrm{Spin}(V)$, viewed as an algebraic group over $\mathbb{Q}$, and write $\bar{G}$ for its image in $\mathrm{SO}(V)$. The group $\mathrm{SL}_2(\mathbb{Q})$ acts on V by conjugation

$$g.X := gXg^{-1}$$

for $X \in V$ and $g \in \mathrm{SL}_2(\mathbb{Q})$, which gives rise to isomorphisms $G \simeq \mathrm{SL}_2$ and $\bar{G} \simeq \mathrm{PSL}_2$. Let $L \subset V(\mathbb{Q})$ be an even lattice of full rank and write L' for the dual lattice of L. Let Γ be a congruence subgroup of $\mathrm{Spin}(L)$ which takes L to itself and acts trivially on the discriminant group L'/L.

Example 2.1. A particularly attractive lattice in V is

$$L = \left\{ \begin{pmatrix} b & c/N \\ a & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice is equal to

$$L' = \left\{ \begin{pmatrix} b & c/N \\ a & -b \end{pmatrix} : a, c \in \mathbb{Z}, b \in \frac{1}{2N}\mathbb{Z} \right\}.$$

We have $L'/L \cong \mathbb{Z}/2N\mathbb{Z}$, the level of L is $4N$ and we can take $\Gamma = \Gamma_0(N)$.

We now recall some facts on vector-valued modular forms and weak Maass forms for the Weil representations. See e.g. [4, 5] for more details.

We let $\mathrm{Mp}_2(\mathbb{R})$ be the two-fold metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ realized as the group of pairs $(g, \phi(g, \tau))$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(g, \tau)$ is a holomorphic square root of the automorphy factor $j(g, \tau) = c\tau + d$, see e.g. [4,5]. Let $\Gamma' \subset \mathrm{Mp}_2(\mathbb{R})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. We denote the standard basis of the group algebra $\mathbb{C}[L'/L]$ by $\{e_h : h \in L'/L\}$. Recall that there is a Weil representation ρ_L of Γ' on the group algebra $\mathbb{C}[L'/L]$, see [4, Section 4] or [5, Chapter 1.1] for explicit formulas.

Let $\Gamma'' \subset \Gamma'$ be a subgroup of finite index. For $k \in \frac{1}{2}\mathbb{Z}$, we let $A_{k,L}(\Gamma'')$ be the space of C^∞ automorphic forms of weight k with respect to ρ_L for Γ''. That is, $A_{k,L}(\Gamma'')$ consists of those C^∞-functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ that satisfy

$$f(\gamma'\tau) = \phi^{2k}(\tau)\rho_L(\gamma', \phi) f(\tau)$$

for $(\gamma', \phi) \in \Gamma''$. Note that the components f_h of f define scalar-valued C^∞ modular forms of weight k for the subgroup $\Gamma'' \cap \Gamma'(N_L)$, where N_L denotes the level of the lattice L and $\Gamma'(N_L)$ is the principal congruence subgroup of level N_L.

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Following [6, Section 3], we call a function $f \in A_{k,L}(\Gamma'')$ a *weak Maass form* of weight k for Γ'' with representation ρ_L if it is an eigenfunction of the hyperbolic Laplacian

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right), \tag{2.2}$$

and if it has at most linear exponential growth at the cusps of Γ''. The latter condition means that there is a $C > 0$ such that for any cusp $s \in \mathbb{P}^1(\mathbb{Q})$ of Γ'' and $(\delta, \phi) \in \Gamma'$ with $\delta\infty = s$ the function $f_s(\tau) = \phi(\tau)^{-2k}\rho_L^{-1}(\delta,\phi) f(\delta\tau)$ satisfies $f_s(\tau) = O(e^{Cv})$ as $v \to \infty$ (uniformly in u, where $\tau = u + iv$).

The function f is called a *harmonic weak Maass form* if it is a weak Maass form with eigenvalue 0 under Δ_k. We write $H_{k,L}(\Gamma'')$ for the space of harmonic weak Maass forms of weight k for Γ'' with representation ρ_L.

Recall that there is a differential operator $\xi_k = 2iv^k\overline{\frac{\partial}{\partial\bar\tau}}$ taking $H_{k,L}(\Gamma'')$ to the space of weakly holomorphic modular forms of 'dual' weight $2-k$ for Γ'' with the dual representation of ρ_L. We let $H^+_{k,L}(\Gamma'')$ be the subspace of those $f \in H_{k,L}(\Gamma'')$ for which $\xi_k(f)$ is a cusp form. Moreover, we let $M^!_{k,L}(\Gamma'')$ be the kernel of ξ_k, that is, the space of weakly holomorphic modular forms for Γ''. Summarizing we have the chain of inclusions

$$M^!_{k,L}(\Gamma'') \subset H^+_{k,L}(\Gamma'') \subset H_{k,L}(\Gamma'') \subset A_{k,L}(\Gamma'').$$

In the case where $\Gamma'' = \Gamma'$ we will drop the Γ'' from the notation and, for instance, simply write $M^!_{k,L}$. If the representation ρ_L is trivial (that is, L is unimodular), we drop the L from the notation.

Example 2.2. We consider the lattice

$$L = \left\{\begin{pmatrix} b & c/N \\ a & -b \end{pmatrix} : a,b,c \in \mathbb{Z}\right\}$$

of level $4N$ from Example 2.1. Then given $g = \sum_{h\in L'/L} g_h e_h$ in $A_{k,L}$, the sum

$$\tilde g(\tau) = \sum_{h\in L'/L} g_h(4N\tau) \tag{2.3}$$

gives a scalar-valued form of weight k for $\Gamma_0(4N)$ satisfying the plus condition, i.e., the n-th Fourier coefficient vanishes unless n is a square modulo $4N$. In fact, if $N = p$ is a prime, and $k \in 2\mathbb{Z} + \frac{1}{2}$, this gives an isomorphism between $M^!_{k,L}$ and the space $M^{+,!}_k(p)$ of scalar-valued weakly holomorphic forms for $\Gamma_0(4p)$ in the Kohnen plus space (see e.g. [4, Example 2.3] and [10, Section 5]).

For an isotropic line ℓ in V, we define the space $W = W_\ell = \ell^\perp/\ell$ which is naturally a unary positive definite quadratic space with the quadratic form $Q(\bar X) = Q(X)$. Then

$$K_\ell = (L \cap \ell^\perp)/(L \cap \ell)$$

defines an even lattice in W. Using [5, Proposition 2.2], it is easy to see that the dual lattice is given by

$$K'_\ell = (L' \cap \ell^\perp)/(L' \cap \ell).$$

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We have the exact sequence

$$0 \to (L' \cap \ell + L \cap \ell^\perp)/L \cap \ell^\perp \to L' \cap \ell^\perp / L \cap \ell^\perp \to K'_\ell / K_\ell \to 0. \tag{2.4}$$

The vector-valued theta function

$$\Theta_{K_\ell}(\tau) = \sum_{\lambda \in K'_\ell} e(Q(\lambda)\tau) \mathfrak{e}_{\lambda + K_\ell}$$

associated to K_ℓ defines a holomorphic modular form in $M_{1/2, K_\ell}$, whose components we denote by $\theta_{K_\ell, \bar{h}}(\tau)$ for $\bar{h} \in K'_\ell / K_\ell$. Recall from [5, Lemma 5.6] (or more generally [27, Theorem 4.1]) that there is a map from vector-valued modular forms for ρ_{K_ℓ} to vector-valued modular forms for ρ_L. Using it, we see that

$$\tilde{\Theta}_{K_\ell}(\tau) = \sum_{\substack{h \in L'/L \\ h \perp \ell}} \theta_{K_\ell, \bar{h}}(\tau) \mathfrak{e}_h \tag{2.5}$$

defines a vector-valued holomorphic modular form in $M_{1/2, L}$. Here $\bar{h}$ denotes the image of h under the map in (2.4). We let $b_\ell(m, h)$ be the (m, h)-th Fourier coefficient of $\tilde{\Theta}_{K_\ell}(\tau)$. Note that for $m > 0$ we have $b_\ell(m, h) = 0$ unless m/N is a square and there exists a vector $X \in L + h$ perpendicular to ℓ of length $Q(X) = m$. In that case we have $b_\ell(m, h) = 1$ if $h \not\equiv -h \mod L$ and 2 otherwise.

3. Cycles and traces

In this section, we introduce the modular curve associated to a given lattice of signature (2, 1) and define cycles and traces in our setting. In particular, we explain in detail how to regularize the periods of weakly holomorphic functions over infinite geodesics.

3.1. Modular curves associated to the orthogonal group SO(2, 1). As in Section 2, we let (V, Q) be space of rational traceless 2×2 matrices together with the quadratic form $Q(X) = -N \det(X)$, see (2.1). We realize the associated hermitian symmetric space as the Grassmannian of negative lines in $V(\mathbb{R})$:

$$D = \{z \subset V(\mathbb{R}) : \dim z = 1 \text{ and } Q|_z < 0\}.$$

We identify D with the complex upper half plane $\mathbb{H}$ as follows, see [18, Section 11]. Let $z_0 \in D$ be the line spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Its stabilizer in $G(\mathbb{R})$ is equal to $K = \mathrm{SO}(2)$. For $z = x + iy \in \mathbb{H}$, we choose $g_z \in G(\mathbb{R})$ such that $g_z i = z$ and put

$$X(z) := \frac{1}{\sqrt{N}} g_z . \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{N} y} \begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix} \in V(\mathbb{R}).$$

We obtain the isomorphism $\mathbb{H} \to D$, $z \mapsto g_z z_0 = \mathbb{R} X(z)$. We also define the quantity

$$R(X, z) = (X, X) + \frac{1}{2}(X, X(z))^2,$$

which is nonnegative, and vanishes exactly when $X \in \mathbb{R} X(z)$.

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For $L \subset V(\mathbb{Q})$ an even lattice let Γ be a congruence subgroup of $\mathrm{Spin}(L)$ which takes L to itself and acts trivially on the discriminant group L'/L. We set $M = \Gamma \backslash D$.

Since V is isotropic, the modular curve M is a non-compact Riemann surface. The group Γ acts on the set $\mathrm{Iso}(V)$ of isotropic lines in V. The cusps of M correspond to the Γ-equivalence classes of $\mathrm{Iso}(V)$, with ∞ corresponding to the isotropic line ℓ_0 spanned by $u_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $\ell \in \mathrm{Iso}(V)$, we pick $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_\ell \ell_0 = \ell$ and set $u_\ell = \sigma_\ell^{-1} u_0$. We let Γ_ℓ be the stabilizer of ℓ in Γ. Then

$$\sigma_\ell^{-1} \overline{\Gamma}_\ell \sigma_\ell = \left\{ \begin{pmatrix} 1 & k\alpha_\ell \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$$

for some $\alpha_\ell \in \mathbb{Z}_{>0}$, the width of the cusp ℓ. There is also a $\beta_\ell \in \mathbb{Q}_{>0}$ such that $\beta_\ell u_\ell$ is a primitive element of $\ell \cap L$. Finally, we write

$$\varepsilon_\ell = \frac{\alpha_\ell}{\beta_\ell}.$$

Note that ε_ℓ does not depend on the choice of σ_ℓ (even if we picked σ_ℓ in $\mathrm{SL}_2(\mathbb{Q})$, see [13, Definition 3.2]).

We compactify M to a compact Riemann surface $\bar{M}$ in the usual way by adding a point for each cusp $\ell \in \Gamma \backslash \mathrm{Iso}(V)$. For every cusp ℓ we choose sufficiently small neighborhoods U_ℓ. We write $q_\ell = e(\sigma_\ell^{-1} z/\alpha_\ell)$ with $z \in U_\ell$ for the local variable (and for the chart) around $\ell \in \bar{M}$. For $T > 0$, we let

$$U_{1/T} = \left\{ w \in \mathbb{C} : |w| < \frac{1}{2\pi T} \right\},$$

and note that for T sufficiently large, the inverse images $q_\ell^{-1}(U_{1/T})$ are disjoint in M. We truncate M by setting

$$M_T = \bar{M} \setminus \coprod_{[\ell] \in \mathrm{Iso}(V)} q_\ell^{-1}(U_{1/T}). \tag{3.1}$$

3.2. Heegner points. Heegner points in M are given as follows, see e.g. [13, 17, 18]. For $X \in V$ of negative length $Q(X) = m < 0$, we put

$$D_X = \mathbb{R}X = \{z \in D : R(X, z) = 0\} \in D.$$

We also write $z_X = D_X$ for the corresponding point in the upper half plane. Via [18, (11.9)] we see

$$R(X, z) = 2|m| \sinh^2(d(z, z_X)) = \frac{|m|}{2\,\mathrm{Im}(z_X)^2 y^2} |z - z_X|^2 |z - \overline{z_X}|^2. \tag{3.2}$$

Here $d(\cdot, \cdot)$ denotes the hyperbolic distance with respect to the standard hyperbolic distance. We note that in the upper half plane we have

$$z_X = \frac{-b}{2a} + \frac{i\sqrt{|d|}}{2|a|}$$

for $X = \left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$ with $Q(X) = Nd < 0$. We set $D_X = \emptyset$ if $Q(X) \geq 0$. The stabilizer G_X of X in $G(\mathbb{R})$ is isomorphic to $\mathrm{SO}(2)$ and for $X \in L'$, the group $\Gamma_X = G_X \cap \Gamma$ is finite. We denote the image of D_X in M, counted with multiplicity $\frac{1}{|\overline{\Gamma}_X|}$, by $Z(X)$.

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For $m \in \mathbb{Q}^\times$ and $h \in L'/L$, the group Γ acts on $L_{m,h} = \{X \in L + h : Q(X) = m\}$ with finitely many orbits. For $m < 0$, we define the *Heegner divisor* of index (m,h) on M by

$$Z(m,h) = \sum_{X \in \Gamma \backslash L_{m,h}} Z(X).$$

For any function f on M, we then define the trace following [31] and [7] by

$$\mathrm{tr}_{m,h}(f) = \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_X|} f(z_X).$$

For the lattice in Example 2.1 with $N = 1$, this gives exactly twice the trace of modular functions defined in the introduction, since our trace counts positive and negative definite binary quadratic forms of discriminant m.

3.3. Geodesics. A vector $X \in V(\mathbb{Q})$ of positive length m defines a geodesic c_X in D via

$$c_X = \{z \in D : z \perp X\} = \{z \in D : (X(z), X) = 0\},$$

see e.g. [17,21,28]. In this situation, we have

$$|(X, X(z))| = 2\sqrt{m}\sinh(d(z, c_X)), \tag{3.3}$$

where $d(z, c_X)$ denotes the hyperbolic distance of z to the geodesic c_X. Hence

$$R(X,z) = 2m\cosh^2(d(z, c_X)).$$

Explicitly, for $X = \left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$, we have

$$c_X = \{z \in D : a|z|^2 + b\,\mathrm{Re}(z) + c = 0\}.$$

We orient the geodesics as follows. For $X = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the geodesic $c_X = \pm(0, i\infty)$ is the imaginary axis with the indicated orientation. The orientation preserving action of $\mathrm{SL}_2(\mathbb{R})$ then induces an orientation for all c_X.

We define the line measure dz_X for c_X by

$$dz_X = \pm\frac{dz}{\sqrt{m}z}$$

for $X = \pm\sqrt{m/N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and then by

$$dz_{g^{-1}X} = d(gz)_X$$

for $g \in \mathrm{SL}_2(\mathbb{R})$. So for $X = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$, we have

$$dz_X = \frac{dz}{az^2 + bz + c}.$$

Note

$$dz_X = -2i\frac{(X, \partial X(z))}{R(X,z)}.$$

Indeed, this holds for $X = \sqrt{m/N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, since then

$$(X, \partial X(z)) = \frac{i\sqrt{m}\bar{z}dz}{y^2} \quad \text{and} \quad R(X,z) = \frac{2m|z|^2}{y^2}.$$

Then the G-equivariance properties of $X(z)$ and $R(X,z)$ imply the claim for general X.

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The stabilizer $\overline{\Gamma}_X$ is either trivial (if the orthogonal complement $X^\perp \subset V$ is isotropic over $\mathbb{Q}$) or infinite cyclic (if $X^\perp$ is non-split over $\mathbb{Q}$). We set $c(X) = \Gamma_X \backslash c_X$, and by slight abuse of notation we use the same symbol for the image of $c(X)$ in M. If $\overline{\Gamma}_X$ is infinite, then $c(X)$ is a closed geodesic in M, while $c(X)$ is an infinite geodesic if $\overline{\Gamma}_X$ is trivial. The last case happens exactly when $Q(X) \in N(\mathbb{Q}^\times)^2$. We define the trace for positive index m and $h \in L'/L$ of a continuous function f on M by

$$\mathrm{tr}_{m,h}(f) = \frac{1}{2\pi} \sum_{X \in \Gamma \backslash L_{m,h}} \int_{c(X)} f(z)\, dz_X.$$

Since

$$\int_{c(X)} f(z)\, dz_X = \int_{c(g^{-1}X)} f(gz)\, dz_{g^{-1}X}, \tag{3.4}$$

for $g \in G$, this is independent of the choice of $X \in \Gamma \backslash L_{m,h}$. Note that a priori the integral only converges if the geodesics are closed, i.e., $m \notin N(\mathbb{Q}^\times)^2$. Otherwise the geodesics $c(X)$ are infinite and $\int_{c(X)} f(z)\, dz_X$ may have to be regularized. We will describe this in the next subsection.

3.4. Infinite geodesics. Assume that X with $Q(X) = m > 0$ gives rise to an infinite geodesic in M, that is, $\overline{\Gamma}_X = 1$. So $c(X) = c_X$. In this section we will show how to regularize the periods of harmonic weak Maass forms over the infinite geodesics. We also define the complementary trace which gives the contribution of the Fourier coefficients of negative index of the holomorphic part of f.

3.4.1. Regularized periods and the central L-value of the j-invariant. We now describe how for $f \in H_0^+(\Gamma)$ we can regularize the period $\int_{c_X} f(z)\, dz_X$. For any isotropic line ℓ, note that $f_\ell(z) := f(\sigma_\ell z)$ can be written as

$$f_\ell = f_\ell^+ + f_\ell^-,$$

where the Fourier expansions of f_ℓ^+ and f_ℓ^- are of the form

$$f_\ell^+(z) = \sum_{n \in \frac{1}{\alpha_\ell}\mathbb{Z}} a_\ell^+(n) e(nz) \quad \text{and} \quad f_\ell^-(z) = \sum_{n \in \frac{1}{\alpha_\ell}\mathbb{Z}_{<0}} a_\ell^-(n) e(n\bar{z}),$$

with $a_\ell^+(n) = 0$ for $n \ll 0$.

Now $X^\perp$ is split over $\mathbb{Q}$, a rational hyperbolic plane spanned by two rational isotropic lines ℓ_X and $\tilde{\ell}_X$. In fact, the geodesic c_X connects the corresponding two cusps (which are not necessarily Γ-inequivalent). We can distinguish these isotropic lines by requiring that ℓ_X represents the endpoint of the oriented geodesic. Note $\tilde{\ell}_X = \ell_{-X}$. We have

$$\sigma_{\ell_X}^{-1} X = \sqrt{m/N} \begin{pmatrix} 1 & -2r \\ 0 & -1 \end{pmatrix}$$

for some $r \in \mathbb{Q}$. Hence the geodesic c_X is explicitly given in $D \simeq \mathbb{H}$ by

$$c_X = \sigma_{\ell_X}\{z \in D : \mathrm{Re}(z) = r\}. \tag{3.5}$$

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We call $r = r_+ = \mathrm{Re}(c_X)$ the *real part* of the geodesic c_X. It depends on the choice of σ_{ℓ_X}. Pick a number $c = c_+ > 0$. We then have (still formally)

$$\sqrt{m}\int_{c_X} f(z)dz_X = \int_{\mathrm{Re}(z)=r_+} f_X(z)\frac{dz}{z-r}$$
$$= \int_{c_+}^{\infty} f_X(iy+r_+)\frac{dy}{y} + \int_{c_-}^{\infty} f_{-X}(iy+r_-)\frac{dy}{y}.$$

Here $f_{\pm X}(z) = f(\sigma_{\ell_{\pm X}} z)$, r_- is the real part of c_{-X} and $c_- = \mathrm{Im}(\sigma_{\ell_{-X}}^{-1}(r+ic_+))$. If we write $r = a/b$ with coprime $a, b \in \mathbb{Z}$ and $b > 0$, then $c_- = (c_+ b^2)^{-1}$.

The extension of the definition (1.10) to the general situation is the following.

Definition 3.1. Let $f \in H_0^+(\Gamma)$ and let c_X be an infinite geodesic connecting two rational cusps in M. Then with the notation as above we set

$$\sqrt{m}\int_{c_X}^{\mathrm{reg}} f(z)dz_X := -a_{\ell_X}^+(0)\log c_+ + \sum_{n\neq 0} a_{\ell_X}^+(n)e^{2\pi i n r_+}\mathcal{E}\mathcal{I}(2\pi n c_+)$$
$$- a_{\ell_{-X}}^+(0)\log c_- + \sum_{n\neq 0} a_{\ell_{-X}}^+(n)e^{2\pi i n r_-}\mathcal{E}\mathcal{I}(2\pi n c_-)$$
$$+ \int_{c_+}^{\infty} f_X^-(iy+r_+)\frac{dy}{y} + \int_{c_-}^{\infty} f_{-X}^-(iy+r_-)\frac{dy}{y}. \tag{3.6}$$

Here $\mathcal{E}\mathcal{I}$ is defined by (1.11). Note that the integrals for f_X^- and f_{-X}^- converge.

Remark 3.2. This definition is independent of the choice of σ_{ℓ_X} and of c_+. While a different σ_{ℓ_X} changes the real part of c_X, it also changes the Fourier coefficients of f in the same way so that the quantities $a_{\ell_X}^+(n)e^{2\pi i n r_+}$ and $a_{\ell_{-X}}^+(n)e^{2\pi i n r_-}$ are in fact invariant. A different choice of c_+, say c_+', changes the first line on the right hand side of (3.6) by

$$-a_{\ell_X}^+(0)(\log c_+' - \log c_+) + \sum_{n\neq 0} a_{\ell_X}^+(n)e^{2\pi i n r_+}(\mathcal{E}\mathcal{I}(2\pi n c_+') - \mathcal{E}\mathcal{I}(2\pi n c_+))$$

and the second by

$$-a_{\ell_{-X}}^+(0)(\log c_-' - \log c_-) + \sum_{n\neq 0} a_{\ell_{-X}}^+(n)e^{2\pi i n r_-}(\mathcal{E}\mathcal{I}(2\pi n c_-') - \mathcal{E}\mathcal{I}(2\pi n c_-)).$$

But both of these expressions are equal to $\int_{c_+'}^{c_+} f_X^+(iy+r_+)\frac{dy}{y}$, but with opposite signs.

We now give a different characterization of the regularized integral. We will need

$$\psi(w) = \frac{\Gamma'(w)}{\Gamma(w)},$$

the digamma function, see [1], for which we have

$$\psi(w) = -\gamma + \sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+w}. \tag{3.7}$$

We set

$$c_X^{c,T} = \{z \in c_X : c \leq \mathrm{Im}(\sigma_{\ell_X} z) \leq T\}.$$

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Theorem 3.3. *Let $f \in H_0^+(\Gamma)$ and assume c_X is a vertical geodesic. Then for any $T_+, T_- > 0$ and the notation as above,*

$$\begin{aligned}\sqrt{m}\int_{c_X}^{\mathrm{reg}} f(z)dz_X = \sqrt{m}&\int_{c_X^{c_+,T_+}} f(z)\,dz_X\\ &-\frac{1}{2}\int_{i\frac{T_+}{\alpha_X}}^{i\frac{T_+}{\alpha_X}+1} f(\alpha_X z + r_+)(\psi(z)+\psi(1-z)+2\log\alpha_X)\,dz\\ &+\sqrt{m}\int_{c_{-X}^{c_-,T_-}} f(z)\,dz_{-X}\\ &-\frac{1}{2}\int_{i\frac{T_-}{\alpha_{-X}}}^{i\frac{T_-}{\alpha_{-X}}+1} f(\alpha_{-X} z + r_-)(\psi(z)+\psi(1-z)+2\log\alpha_{-X})\,dz\\ &+\int_{c_+}^{\infty} f_X^-(iy+r_+)\frac{dy}{y} + \int_{c_-}^{\infty} f_{-X}^-(iy+r_-)\frac{dy}{y}.\end{aligned}$$

In particular for $T_+ = c_+$ and $T_- = c_-$ and $f \in M_0^!(\Gamma)$,

$$\begin{aligned}\sqrt{m}\int_{c_X}^{\mathrm{reg}} f(z)dz_X = &-\frac{1}{2}\int_{i\frac{c_+}{\alpha_X}}^{i\frac{c_+}{\alpha_X}+1} f(\alpha_X z + r_+)(\psi(z)+\psi(1-z)+2\log\alpha_X)\,dz\\ &-\frac{1}{2}\int_{i\frac{c_-}{\alpha_{-X}}}^{i\frac{c_-}{\alpha_{-X}}+1} f(\alpha_{-X} z + r_-)(\psi(z)+\psi(1-z)+2\log\alpha_{-X})\,dz.\end{aligned}$$

Proof. Since the general case is similar, for simplicity, we assume that c_X is the imaginary axis and also $f \in M_0^!(\Gamma)$. We first show that the above formulas are independent of the choice of $T_\pm$. For that we also assume for the moment $\alpha_X = 1$, again for simplicity. Pick another $T_1 > 0$. By Cauchy's theorem we have

$$\begin{aligned}(3.8)\qquad -\int_{iT}^{iT+1} f(z)\psi(z)\,dz = &-\int_{iT_1+1}^{iT+1} f(z)\psi(z)\,dz - \int_{iT_1}^{iT_1+1} f(z)\psi(z)\,dz\\ &+\int_{iT_1}^{iT} f(z)\psi(z)\,dz.\end{aligned}$$

For the first integral on the right hand side we see using $\psi(z+1) = \psi(z) + \frac{1}{z}$ that

$$(3.9)\qquad -\int_{iT_1+1}^{iT+1} f(z)\psi(z)\,dz = -\int_{iT_1}^{iT} f(z)\psi(z)\,dz - \int_{iT_1}^{iT} f(z)\frac{dz}{z}.$$

We also have

$$(3.10)\qquad \int_{c_X^{c_+,T}} f(z)\,dz_X = \int_{ic_+}^{iT} f(z)\frac{dz}{\sqrt{m}z}.$$

Then by (3.10), (3.8), and (3.9) we conclude, recalling that $\alpha_X = 1$, that

$$\int_{c_X^{c_+,T}} f(z)\,dz_X - \int_{iT}^{iT+1} f(z)\psi(z)\frac{dz}{\sqrt{m}} = \int_{c_X^{c_+,T_1}} f(z)\,dz_X - \int_{iT_1}^{iT_1+1} f(z)\psi(z)\frac{dz}{\sqrt{m}}.$$

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The same holds for $\psi(z)$ replaced by $\psi(1-z)$. This shows the independence of the choice of T. Now assume $T = c_+ = c$. With $\alpha = \alpha_X$ arbitrary, we first have

$$\begin{aligned}
&- a_{\ell_X}(0) \int_{ic/\alpha}^{ic/\alpha+1} (\psi(z) + \psi(1-z))\, dz \\
&\quad = -a_{\ell_X}(0)\left[\log\left(\frac{\Gamma(1+\frac{ic}{\alpha})}{\Gamma(\frac{ic}{\alpha})}\right) - \log\left(\frac{\Gamma(1-(1+\frac{ic}{\alpha}))}{\Gamma(1-\frac{ic}{\alpha})}\right)\right] \\
&\quad = -2a_{\ell_X}(0)\log\left(\frac{c}{\alpha}\right).
\end{aligned}$$

Now consider $f_0(z) = f(z) - a_{\ell_X}(0)$. Then from (3.7) we see

$$\begin{aligned}
\text{(3.11)}\qquad &\int_{ic/\alpha}^{ic/\alpha+1} f_0(\alpha z)(\psi(z) + \psi(1-z))\, dz \\
&\quad = -\int_{ic/\alpha}^{ic/\alpha+\infty} f_0(\alpha z)\frac{dz}{z} + \int_{ic/\alpha}^{ic/\alpha-\infty} f_0(\alpha z)\frac{dz}{z}.
\end{aligned}$$

Plugging in the Fourier expansion of f_0, we obtain

$$\begin{aligned}
&-\int_{ic/\alpha}^{ic/\alpha+\infty} f_0(\alpha z)\frac{dz}{z} + \int_{ic/\alpha}^{ic/\alpha-\infty} f_0(\alpha z)\frac{dz}{z} \\
&\quad = -\sum_{n\neq 0} a_{\ell_X}(n)\left(\int_{ic}^{ic+\infty} e^{2\pi i n z}\frac{dz}{z} + \overline{\int_{ic}^{ic+\infty} e^{2\pi i n z}\frac{dz}{z}}\right).
\end{aligned}$$

According to [1, equations (5.1.30) and (5.1.31)], we have

$$\int_{ic}^{ic+\infty} e^{2\pi i n z}\frac{dz}{z} = \begin{cases} E_1(2\pi n c) & \text{if } n > 0, \\ -\operatorname{Ei}(2\pi|n|c) - i\pi & \text{if } n < 0. \end{cases}$$

So, finally,

$$-\frac{1}{2}\int_{ic}^{ic+1} f(z)(\psi(z) + \psi(1-z) + 2\log\alpha)\, dz = -a_{\ell_X}(0)\log(c) + \sum_{n\neq 0} a_{\ell_X}(n)\mathcal{E}\mathcal{I}(2\pi n c).$$

Carrying out the same analysis for the other cusp ℓ_{-X} completes the proof of the theorem. □

Remark 3.4. We consider $j_1(z) \in M_0^!(\mathrm{SL}_2(\mathbb{Z}))$. Then applying Theorem 3.3 gives

$$\int_0^{\infty,reg} j_1(iy)\frac{dy}{y} = -\int_i^{i+1} j_1(z)(\psi(z) + \psi(1-z))\, dz.$$

This is exactly Zagier's regularization for the 'central L-value of j_1'. He arrived to this formula by the following heuristic considerations. We need to give a meaning to the expression

$$2\int_0^i j_1(z)\frac{dz}{z}.$$

We deform the path of integration to the semicircle to the left (respectively right) of the imaginary axis starting at 0 and ending at i. Under the transformation $z \to -\frac{1}{z}$ this path becomes

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the horizontal half line in the upper half plane beginning at ∞ (respectively $-\infty$) ending at i. Hence we obtain

$$\int_i^{i+\infty} j_1(z)\frac{dz}{z} + \int_i^{i-\infty} j_1(z)\frac{dz}{z}.$$

But now these integrals converge, and by (3.11) we obtain

$$-\int_i^{i+1} j_1(z)(\psi(z)+\psi(1-z))\,dz = -2\operatorname{Re}\left(\int_i^{i+1} j_1(z)\psi(z)\,dz\right).$$

Here we used that the Fourier coefficients of j_1 are real. In fact, one can show that this is equal to $-2\operatorname{Re}\int_{\rho^2}^{\rho} j_1(z)\psi(z)\,dz$, where $\rho = e^{2\pi i/6}$.

Remark 3.5. We work out the trace $\operatorname{tr}_{m,0}(1)$ for the constant function 1 when $\frac{m}{N}$ is an integral square. Then

$$\operatorname{tr}_{m,0}(1) = -\frac{1}{\pi\sqrt{\frac{m}{N}}} \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \sum_{k=1}^{2\sqrt{\frac{m}{N}}\varepsilon_\ell} \log\frac{\gcd\left(k\beta_\ell, 2\sqrt{\frac{m}{N}}\right)}{2\sqrt{\frac{m}{N}}}.$$

Here $\varepsilon_\ell = \frac{\alpha_\ell}{\beta_\ell}$ is defined in Section 3.1. Indeed, we can sort the infinite geodesics C_m by the cusps ℓ to which they go. By [13, Lemma 3.7], there exist $2\sqrt{m/N}\varepsilon_\ell$ many X in $\Gamma\backslash L_{m,0}$ such that the corresponding geodesics c_X all end in ℓ. They have real part

$$\frac{k\beta_\ell}{2\sqrt{\frac{m}{N}}}, \qquad k = 1,\dots,2\sqrt{\frac{m}{N}}\varepsilon_\ell.$$

The claim then easily follows from taking $c_+ = 1$ in Definition 3.1.

3.4.2. Complementary trace. Assume that X with $Q(X) > 0$ gives rise to an infinite geodesic, that is, $\overline{\Gamma}_X = 1$. Let $f \in H_0^+(\Gamma)$ be a weak Maass form with holomorphic Fourier coefficients $a_\ell^+(n)$. Then we define its *complementary trace* for $m \in N(\mathbb{Q}^\times)^2$ and $h \in L'/L$ by

$$\operatorname{tr}^c_{m,h}(f) = \sum_{X\in\Gamma\backslash L_{m,h}} \sum_{n<0} a^+_{\ell_X}(n)e^{2\pi i\operatorname{Re}(c(X))n} + \sum_{n<0} a^+_{\ell_{-X}}(n)e^{2\pi i\operatorname{Re}(c(-X))n}.$$

Note that in [7] this quantity is denoted by $\operatorname{tr}_{m,h}(f)$. We have (see [7, Proposition 4.7])

$$\begin{aligned}\operatorname{tr}^c_{m,h}(f) = 2\sqrt{m/N} \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \varepsilon_\ell \Bigg[\delta_\ell(m,h) &\sum_{\substack{n\in\frac{2}{\beta_\ell}\sqrt{\frac{m}{N}}\mathbb{Z}\\ n<0}} a^+_\ell(n)e^{2\pi i r_+ n} \\ &+ \delta_\ell(m,-h) \sum_{\substack{n\in\frac{2}{\beta_\ell}\sqrt{\frac{m}{N}}\mathbb{Z}\\ n<0}} a^+_\ell(n)e^{2\pi i r_- n}\Bigg].\end{aligned}$$

Here $\delta_\ell(m,h) = 1$ if the (m,h)-th Fourier coefficient $b_\ell(m,h)$ of $\tilde{\Theta}_{K_\ell}(\tau)$ is nonzero, that is, if there exists a vector $X \in L_{m,h}$ such that c_X ends at the cusp ℓ. In that case $r_\pm$ is the real part of any such X. In particular, $\operatorname{tr}^c_{m,h}(f) = 0$ for $m \gg 0$.

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3.5. Average values of harmonic weak Maass forms. We define the regularized 'average value' of a (suitable) function f on M by

$$\int_M^{\mathrm{reg}} f(z)\,d\mu(z) = \lim_{T\to\infty} \int_{M_T} f(z)\,d\mu(z) \tag{3.12}$$

as in [7, (4.6)]. By [7, Remark 4.9] we have for weakly holomorphic f that

$$\int_M^{\mathrm{reg}} f(z)\,d\mu(z) = -8\pi \sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \alpha_\ell \sum_{N\in\mathbb{Z}_{\geq 0}} a_\ell(-N)\sigma_1(N). \tag{3.13}$$

Here $\sigma_1(0) = -\frac{1}{24}$. The formula also holds for $f \in H_0^+(\Gamma)$. Indeed, we let

$$\mathcal{E}_2(z) = -\frac{3}{\pi y} - 24\sum_{n=0}^{\infty} \sigma_1(n) e^{2\pi i n z}$$

be the (non-holomorphic) Eisenstein series $\mathcal{E}_2(z)$ of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\bar\partial(\mathcal{E}_2(z)\,dz) = -\frac{3}{\pi}\,d\mu(z).$$

Hence by Stokes's theorem we obtain

$$\int_M^{\mathrm{reg}} f(z)\,d\mu(z) = -\frac{\pi}{3}\lim_{T\to\infty}\int_{\partial M_T} f(z)\mathcal{E}_2(z)\,dz + \int_M (\bar\partial f(z))\mathcal{E}_2(z)\,dz.$$

The first term gives (3.13), while the second vanishes as the Petersson scalar product of the cusp form $\xi_0(f)$ against an Eisenstein series.

We define as in [7] the trace of index $(0,h)$ by

$$\mathrm{tr}_{0,h}(f) = -\delta_{h,0}\frac{1}{2\pi}\int_M^{\mathrm{reg}} f(z)\,d\mu(z).$$

Here $\delta_{h,0}$ is Kronecker delta. For $f = 1$, we also write

$$\mathrm{vol}(M) = -\frac{1}{2\pi}\int_M d\mu(z).$$

4. The main result

We are now ready to state the main result of this paper. It will be proved using the regularized theta lift in Sections 7 and 8.

Theorem 4.1. *Let $h \in L'/L$. Let $f \in H_0^+(\Gamma)$ be a weak Maass form and assume that the constant coefficients $a_\ell^+(0)$ vanish at all cusps ℓ. Then the generating series*

$$\begin{aligned} H_h(\tau, f) := {}& -2\sqrt{v}\,\mathrm{tr}_{0,h}(f) \\ & + \sum_{m<0} \mathrm{tr}_{m,h}(f)\frac{\mathrm{erfc}(2\sqrt{\pi|m|v})}{2\sqrt{|m|}} e(m\tau) + \sum_{m>0}\mathrm{tr}_{m,h}(f)e(m\tau) \\ & + 2\sum_{m>0}\mathrm{tr}^c_{Nm^2,h}(f)\left(\int_0^{\sqrt{v}} e^{4\pi N m^2 w^2}\,dw\right)e(Nm^2\tau) \end{aligned}$$

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defines the h-component of a weak Maass form of weight $1/2$ for the representation ρ_L. If f has non-zero constant coefficients, then one has to add

$$-\frac{1}{\sqrt{N}\pi}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} a_\ell^+(0)\varepsilon_\ell\Big[\frac{1}{2}\big(\log(4\beta_\ell^2\pi v)+\gamma+\psi(k_\ell/\beta_\ell)+\psi(1-k_\ell/\beta_\ell)\big) + \sum_{m>0} b_\ell(Nm^2,h)\mathcal{F}(2\sqrt{\pi v N}m)e(Nm^2\tau)\Big]$$

to the generating series. Here $b_\ell(m,h)$ denotes the (m,h)-th coefficient of $\tilde{\Theta}_{K_\ell}(\tau)$ defined in (2.5)*. For the quantities ε_ℓ and β_ℓ, see Section* 3.1*, and k_ℓ is defined by*

$$\ell\cap(L+h)=\mathbb{Z}\beta_\ell u_\ell+k_\ell u_\ell \quad \textit{and} \quad 0\le k_\ell<\beta_\ell.$$

Furthermore, we (formally) set $\psi(0)=-\gamma$, which is justified since $-\gamma$ is the constant term of the Laurent expansion of ψ at 0*. Finally, we have set*

$$\mathcal{F}(t):=\log t-\sqrt{\pi}\left(\int_0^t e^{w^2}\operatorname{erfc}(w)\,dw\right)+\frac{1}{2}\log(2)+\frac{1}{4}\gamma,$$

where

$$\operatorname{erfc}(w)=\frac{2}{\sqrt{\pi}}\int_w^\infty e^{-t^2}\,dt$$

is the complementary error function. Finally,

$$\Delta_\tau H_h(\tau,f)=-\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)}\frac{a_\ell^+(0)\varepsilon_\ell}{4\sqrt{N}\pi}\theta_{K_{\ell,\bar{h}}}(\tau).$$

As a special case we consider the constant function $f=1$.

Theorem 4.2. *Let $h\in L'/L$. Then*

$$\begin{aligned}
H_h(\tau,1)={}&-2\sqrt{v}\operatorname{vol}(M)+\sum_{m<0}\deg Z(m,h)\frac{\operatorname{erfc}(2\sqrt{\pi|m|v})}{2\sqrt{|m|}}e(m\tau)\\
&+\sum_{\substack{m>0\\ m\notin N(\mathbb{Q}^\times)^2}}\Big(\sum_{X\in\Gamma\backslash L_{m,h}}\operatorname{length}(c(X))\Big)e(m\tau)+\sum_{m>0}\operatorname{tr}_{Nm^2,h}(1)q^{Nm^2}\\
&-\frac{1}{\sqrt{N}\pi}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)}\varepsilon_\ell\sum_{m>0}b_\ell(Nm^2,h)(\mathcal{F}(2\sqrt{\pi v N}m))e(Nm^2\tau)\\
&-\frac{1}{2\sqrt{N}\pi}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)}\varepsilon_\ell\big[\log(4\beta_\ell^2\pi v)+\gamma+\psi(k_\ell/\beta_\ell)+\psi(1-k_\ell/\beta_\ell)\big]
\end{aligned}$$

defines the h-component of a weak Maass form for ρ_L of weight $1/2$.

Remark 4.3. The special functions $\mathcal{F}$ above and α in (1.7) differ by a constant, since both map to $\beta_{3/2}$ under the differential operator

$$\xi_{1/2}=2iv^{1/2}\overline{\frac{\partial}{\partial\bar{\tau}}}.$$

This constant is absorbed by the undefined term $\operatorname{tr}_{Nm^2}(1)$ in (1.7).

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Remark 4.4. Applying $\xi_{1/2}$ to $H_h(\tau, f)$ we recover the generating series for the traces of modular functions over CM points obtained in [7] (and [13] for $f = 1$) which in turn generalized the results of Zagier [30, 31].

Example 4.5. We recover the theorems in the introduction by considering the lattice of Example 2.1 for $N = p$ using Example 2.2. Alternatively, one can consider for $N = 1$ the lattice

$$L = \left\{ \begin{pmatrix} b & 2c \\ -2ap & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

and employ the same arguments as in [7, Section 6].

Example 4.6. We consider for $N = 1$ the lattice L of Example 2.1. Let

$$h \in L'/L \cong \mathbb{Z}/2\mathbb{Z}$$

be the non-trivial element. Then for $m \in \mathbb{Z}_{>0}$ we have

$$\mathrm{tr}_{-m,L}(1) = 2H(4m) \quad \text{and} \quad \mathrm{tr}_{-m/4,L}(1) + \mathrm{tr}_{-m/4,L+h}(1) = 2H(m),$$

where $H(m)$ is the Kronecker–Hurwitz class number, see e.g. [13, Section 3]. We let $r_3(m)$ be the representation number of m as the sum of three squares. Then the famous class number relation states $r_3(m) = 12(H(4m) - 2H(m))$. Hence if we define

$$H(\tau) := 6\big(2H_L(4\tau, 1) + 2H_{L+h}(4\tau, 1) - H_L(\tau, 1)\big),$$

we obtain

$$\xi_{1/2} H(\tau) = \theta^3(\tau).$$

5. Theta series and the regularized theta lift

In this section we define regularized theta lifts of automorphic functions with singularities on M against the theta function $\Theta_L(\tau, z, \varphi_0)$.

5.1. Some Schwartz functions. We consider the standard Gaussian φ_0 on $V(\mathbb{R})$,

$$\varphi_0(X, z) = e^{-\pi(X,X)_z},$$

where $(X, X)_z$ is the majorant associated to $z \in D$ which is given by

$$(X, X)_z = (X, X) + (X, X(z))^2. \tag{5.1}$$

Hence $(X, X)_z = -(X, X) + 2R(X, z)$. Recall that the Schwartz function φ_0 has weight $1/2$ under the Weil representation acting on the space of Schwartz functions on $V(\mathbb{R})$, see e.g. [28]. Accordingly, for $\tau = u + iv \in \mathbb{H}$, we define

$$\varphi_0(X, \tau, z) = \sqrt{v}\varphi_0(\sqrt{v}X, z)e^{\pi i (X,X)u} = \sqrt{v}e^{\pi i (X,X)_{\tau,z}},$$

where

$$(X, X)_{\tau,z} = u(X, X) + iv(X, X)_z = (X, X)\bar{\tau} + 2ivR(X, z).$$

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To distinguish between the Laplacians acting on functions in the two variables $z \in D$ and $\tau \in \mathbb{H}$, we often write

$$\Delta_z = \Delta_{0,z} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

for the hyperbolic Laplacian of weight 0 on D, and write

$$\Delta_\tau = \Delta_{1/2,\tau}$$

for the hyperbolic Laplacian on $\mathbb{H}$ of weight $1/2$ as in (2.2). We have

$$-\Delta_{1/2,\tau} = L_{5/2}R_{1/2} + \frac{1}{2} = R_{-3/2}L_{1/2},$$

where

$$R_k = 2i\frac{\partial}{\partial\tau} + kv^{-1},$$
$$L_k = -2iv^2\frac{\partial}{\partial\bar{\tau}}$$

are the weight k raising and lowering operators acting on functions on $\mathbb{H}$. The following lemma is well known, see e.g. [28], [17, p. 205, equation (2.10)].

Lemma 5.1. *We have*

$$\Delta_\tau\varphi_0(X,\tau,z) = \frac{1}{4}\Delta_z\varphi_0(X,\tau,z).$$

We define another Schwartz function by

$$\varphi_1(X,\tau,z) := -\frac{1}{\pi}L_{\frac{1}{2}}\varphi_0(X,\tau,z). \tag{5.2}$$

Hence φ_1 has weight $-3/2$. The next lemma is then immediate.

Lemma 5.2. *We have*

$$\Delta_z\varphi_0(X,\tau,z) = 4\pi R_{-3/2}\varphi_1(X,\tau,z).$$

Remark 5.3. The Schwartz function $\varphi_1^V = \varphi_1$ associated to the space V of signature $(2,1)$ is very closely related to the Kudla–Millson Schwartz form $\varphi_{KM}^{V^-}$ for the space V^-, which is the vector space V together with the quadratic form $-Q(X)$ of signature $(1,2)$ (see also [7, Section 7]). The form $\varphi_{KM}^{V^-}$ has weight $3/2$ and

$$\varphi_{KM}^{V^-}(X,\tau,z) = v^{-3/2}\overline{\varphi_1(X,\tau,z)}\,d\mu(z).$$

Here $d\mu(z) = \frac{dx\,dy}{y^2}$ is the invariant volume form on D. Moreover, we see that

$$\varphi_{KM}^{V^-}(X,\tau,z) = -\frac{1}{\pi}\xi_{1/2}\varphi_0(X,\tau,z)\,d\mu(z),$$

where

$$\xi_k f = v^{k-2}\overline{L_k f} = R_{-k}v^k\overline{f}.$$

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5.2. Theta series and theta lifts. We let φ be the Schwartz function φ_0 or φ_1 on $V(\mathbb{R})$ of weight k (equal to $1/2$ or $-3/2$). Then for $h \in L'/L$, we define a theta series by

$$\theta_h(\tau, z, \varphi) = \sum_{X \in h+L} \varphi(X, \tau, z).$$

In the variable z it is Γ-invariant and therefore descends to a function on M. We also define a vector-valued theta series by

$$\Theta_L(\tau, z, \varphi) = \sum_{h \in L'/L} \theta_h(\tau, z, \varphi) \mathfrak{e}_h.$$

As a function of $\tau \in \mathbb{H}$, we have that $\Theta_L(\tau, z, \varphi) \in A_{k,L}$, see e.g. [4].

We let $f(z)$ be a Γ-invariant function on D. We define the theta lift of f by

$$\begin{aligned} I(\tau, f) &= \int_M f(z)\Theta_L(\tau, z, \varphi_0)\, d\mu(z) \\ &= \sum_{h \in L'/L} \left(\int_M f(z)\theta_h(\tau, z, \varphi_0)\, d\mu(z) \right) \mathfrak{e}_h, \end{aligned} \tag{5.3}$$

where $d\mu(z) = \frac{dx\,dy}{y^2}$ is the invariant volume form on D. We also write

$$I_h(\tau, f) = \int_M f(z)\theta_h(\tau, z, \varphi_0) d\mu(z) \tag{5.4}$$

for the individual components. If f is of sufficiently rapid decay at the cusps, the theta integral converges and defines a (in general non-holomorphic) modular form on the upper half plane of weight $1/2$ of type ρ_L. In fact, for f a Maass cusp form, the lift was considered by Katok–Sarnak [17]. In the present paper, we are particularly interested in the case when f is *not* of rapid decay at the cusps. Then the theta integral typically does not converge and needs to be regularized. We will carry this out in the remainder of this section. In Sections 7 and 8 will show that $I(\tau, f)$ for $f \in H_0^+(\Gamma)$ will give the generating series for the traces given in Section 4.

5.3. The growth of the theta kernel. The growth of the theta functions $\Theta_L(\tau, z, \varphi)$ near the cusps of M is given as follows.

Proposition 5.4. *As $y \to \infty$,*

(i) $\Theta_L(\tau, \sigma_\ell z, \varphi_0) = y \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) + O(e^{-Cy^2})$.

In particular, if $\ell^\perp \cap (L + h) = \emptyset$, then $\theta_h(\tau, \sigma_\ell z, \varphi_0) = O(e^{-Cy^2})$.

(ii) $\Theta_L(\tau, \sigma_\ell z, \varphi_1) = O(e^{-Cy^2})$.

(iii) $\Theta_L(\tau, \sigma_\ell z, \Delta_z \varphi_0) = O(e^{-Cy^2})$.

In particular, $\theta_h(\tau, z, \varphi_1)$ and $\theta_h(\tau, z, \Delta_z\varphi_0)$ are 'square exponentially' decreasing at all cusps of M.

Proof. This follows from a very special case of [4, Theorem 5.2]. For φ_0 also see [5, Theorem 2.4], and for φ_1 in view of Remark 5.3 see the proof of [13, Proposition 4.1] and [7, Proposition 4.1]. For convenience we sketch the proof.

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(i) Since

$$\Theta_L(\tau, \sigma_\ell z, \varphi) = \Theta_{\sigma_\ell^{-1} L}(\tau, z, \varphi),$$

it is enough to consider the cusp $\infty = \ell_0$. A primitive norm 0 vector of $\sigma_\ell^{-1} L$ in ℓ_0 is given by

$$B_\ell := \begin{pmatrix} 0 & \beta_\ell \\ 0 & 0 \end{pmatrix}.$$

We may now apply [4, Theorem 5.2] for the lattice $\sigma_\ell^{-1} L$, the Schwartz function φ_0, and the primitive norm 0 vector B_ℓ to obtain the assertion.

(ii) Since $\varphi_1 = -\frac{1}{\pi} L_{1/2} \varphi_0$ and since $\tilde{\Theta}_{K_\ell}(\tau)$ is holomorphic, we obtain (ii) by applying the lowering operator $L_{1/2}$ to (i).

(iii) This follows from (ii) by applying the raising operator $R_{-3/2}$ to $\Theta_L(\tau, z, \varphi_1)$. □

5.4. Regularization using truncated fundamental domains. Following an idea of Borcherds [4] and Harvey–Moore [15], we regularize the theta integral by integrating over a truncated fundamental domain and then taking a limit.

If $h(s)$ is a meromorphic function in a neighborhood of $s_0 \in \mathbb{C}$, we shall denote by $\mathrm{CT}_{s=s_0}[h(s)]$ the constant term in the Laurent expansion of h at $s = s_0$. Let

$$\mathcal{F} = \{z \in \mathbb{H} : |x| \le 1/2 \text{ and } |z| \ge 1\}$$

be the standard fundamental domain for the action of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ on the upper half plane. For a positive integer a we put

$$\mathcal{F}^a := \bigcup_{j=0}^{a-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \mathcal{F}.$$

As before, let $\Gamma \subset \Gamma(1)$ be a congruence subgroup. Recall that for $\ell \in \mathrm{Iso}(V)$ we choose $\sigma_\ell \in \Gamma(1)$ such that $\sigma_\ell \ell_0 = \ell$, and α_ℓ denotes the width of the cusp ℓ, see Section 3.1.

Lemma 5.5. *We have the disjoint left coset decomposition*

$$\bar{\Gamma}(1) = \bigcup_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \bigcup_{j \in \mathbb{Z}/\alpha_\ell \mathbb{Z}} \bar{\Gamma} \sigma_\ell \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}.$$

Consequently, a fundamental domain or the action of Γ on D is given by

$$\mathcal{F}(\Gamma) = \bigcup_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \sigma_\ell \mathcal{F}^{\alpha_\ell}. \tag{5.5}$$

Moreover, if f is of rapid decay, the theta integral (5.3) is given by

$$I(\tau, f) = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \int_{\mathcal{F}^{\alpha_\ell}} f(\sigma_\ell z) \Theta_L(\tau, \sigma_\ell z, \varphi_0)\, d\mu(z). \tag{5.6}$$

Now assume that f is a function on M which is not necessarily decaying at the cusps. For $T > 0$, we truncate $\mathcal{F}^a$ and put $\mathcal{F}^a_T := \{z \in \mathcal{F}^a : y \le T\}$.

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Definition 5.6. We define the regularized theta lift of f by

$$I^{\mathrm{reg}}(\tau, f) = \mathrm{CT}_{s=0}\big[I^{\mathrm{reg}}(\tau, s, f)\big],$$

where

$$I^{\mathrm{reg}}(\tau, s, f) = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Theta_L(\tau, \sigma_\ell z, \varphi_0) y^{-s} \, d\mu(z). \tag{5.7}$$

Here s is an additional complex variable. If f is rapidly decaying at the cusps, then it is easily seen that the regularized theta lift agrees with the classical lift (5.6). However, as we will now see, the regularized lift makes sense for a much wider class of functions f.

Proposition 5.7. *Let f be a weak Maass form of weight zero for Γ with eigenvalue $\lambda = s'(1-s')$. Denote the constant term of the Fourier expansion of f at the cusp ℓ by $A_\ell y^{s'} + B_\ell y^{1-s'}$ with constants $A_\ell, B_\ell \in \mathbb{C}$. Then $I^{\mathrm{reg}}(\tau, s, f)$ converges locally uniformly in s for* $\mathrm{Re}(s) > \max(\mathrm{Re}(s'), 1-\mathrm{Re}(s'))$ *and defines an element of $A_{1/2,L}$. It has a meromorphic continuation to the whole s-plane, which is holomorphic in s up to first order poles at $s = s'$ and $s = 1-s'$. The function*

$$I^{\mathrm{reg}}(\tau, s, f) - \frac{1}{\sqrt{N}} \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \varepsilon_\ell \tilde{\Theta}_{K_\ell}(\tau) \left(\frac{A_\ell}{s-s'} + \frac{B_\ell}{s+s'-1} \right)$$

has a holomorphic continuation to all $s \in \mathbb{C}$. Moreover, the regularized theta lift $I^{\mathrm{reg}}(\tau, f)$ defines an element of $A_{1/2,L}$.

Proof. It suffices to show that for any $\ell \in \Gamma \backslash \mathrm{Iso}(V)$, the integral

$$I_\ell^{\mathrm{reg}}(\tau, s, f) = \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Theta_L(\tau, \sigma_\ell z, \varphi_0) y^{-s} \, d\mu(z) \tag{5.8}$$

converges for $\mathrm{Re}(s) > \max(\mathrm{Re}(s'), 1-\mathrm{Re}(s'))$ and has a meromorphic continuation in s with the appropriate poles and residues. In view of Proposition 5.4, we split up the integral as follows:

$$\begin{aligned} I_\ell^{\mathrm{reg}}(\tau, s, f) = \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Big(\Theta_L(\tau, \sigma_\ell z, \varphi_0) - y \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \Big) y^{-s} \, d\mu(z) \\ + \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) y^{1-s} \, d\mu(z). \end{aligned}$$

Because of Proposition 5.4, the function in the integral of the first summand is of square exponential decay as $y \to \infty$. Therefore the first summand converges for all $s \in \mathbb{C}$ and defines a holomorphic function of s.

For the second summand we split up the integral as

$$\begin{aligned} \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \frac{\tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}\beta_\ell} y^{1-s} \, d\mu(z) = \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \int_{\mathcal{F}_1^{\alpha_\ell}} f(\sigma_\ell z) y^{1-s} \, d\mu(z) \\ + \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \int_{y=1}^{\infty} \int_{x=0}^{\alpha_\ell} f(\sigma_\ell z) \, dx \, \frac{dy}{y^{1+s}}. \end{aligned}$$

The first summand on the right hand side is an integral over a compact domain. It converges for all s and defines a holomorphic function in s. For the second summand on the right hand

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side we use the Fourier expansion of f at the cusp ℓ,

$$f(z) = \sum_{n \in \mathbb{Z}} a_\ell(n, y) e\left(\frac{nx}{\alpha_\ell}\right).$$

We find that it is given by

$$\frac{\alpha_\ell}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \int_{y=1}^{\infty} a_\ell(0, y) \frac{dy}{y^{1+s}}. \tag{5.9}$$

If we write

$$a_\ell(0, y) = A_\ell y^{s'} + B_\ell y^{1-s'},$$

we see that the integral exists for $\mathrm{Re}(s) > \max(\mathrm{Re}(s'), 1 - \mathrm{Re}(s'))$, and it is equal to

$$\frac{\alpha_\ell}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \left(\frac{A_\ell}{s - s'} + \frac{B_\ell}{s + s' - 1} \right).$$

Using $\varepsilon_\ell = \alpha_\ell / \beta_\ell$ and putting together the contributions of the different cusps ℓ, we obtain the assertion. □

In the next proposition we give a formula for $I^{\mathrm{reg}}(\tau, f)$ as a limit, not involving the additional parameter s.

Proposition 5.8. *Let f be a weak Maass form of weight zero for Γ with eigenvalue $\lambda = s'(1 - s')$. Denote the constant term of the Fourier expansion of f at the cusp ℓ by $A_\ell y^{s'} + B_\ell y^{1-s'}$ with constants $A_\ell, B_\ell \in \mathbb{C}$. If $s' \neq 0, 1$, then $I^{\mathrm{reg}}(\tau, f)$ is equal to*

$$\sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \lim_{T \to \infty} \left[\int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Theta_L(\tau, \sigma_\ell z, \varphi_0) d\mu(z) - \left(A_\ell \frac{T^{s'}}{s'} + B_\ell \frac{T^{1-s'}}{1 - s'} \right) \frac{\varepsilon_\ell \tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}} \right].$$

If $s' = 0$, then the same formula holds when $\frac{T^{s'}}{s'}$ is replaced by $\log(T)$. If $s' = 1$, then the same formula holds when $\frac{T^{1-s'}}{1-s'}$ is replaced by $\log(T)$.

Proof. For a cusp ℓ, we let $I_\ell^{\mathrm{reg}}(\tau, f) = \mathrm{CT}_{s=0}[I_\ell^{\mathrm{reg}}(\tau, s, f)]$. Then we have

$$I^{\mathrm{reg}}(\tau, f) = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} I_\ell^{\mathrm{reg}}(\tau, f),$$

and it suffices to show that $I_\ell^{\mathrm{reg}}(\tau, f)$ is given by

$$\lim_{T \to \infty} \left[\int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Theta_L(\tau, \sigma_\ell z, \varphi_0) \, d\mu(z) - \left(A_\ell \frac{T^{s'}}{s'} + B_\ell \frac{T^{1-s'}}{1 - s'} \right) \frac{\varepsilon_\ell \tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}} \right].$$

The proof of Proposition 5.7 shows that $I_\ell^{\mathrm{reg}}(\tau, f)$ is equal to

$$\begin{aligned} &\lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \left(\Theta_L(\tau, \sigma_\ell z, \varphi_0) - y \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \right) d\mu(z) \\ &\quad + \frac{1}{\sqrt{N}\beta_\ell} \tilde{\Theta}_{K_\ell}(\tau) \int_{\mathcal{F}_1^{\alpha_\ell}} f(\sigma_\ell z) y \, d\mu(z) \\ &\quad + \frac{\varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau) \, \mathrm{CT}_{s=0} \left[\int_{y=1}^{\infty} (A_\ell y^{s'} + B_\ell y^{1-s'}) \frac{dy}{y^{1+s}} \right]. \end{aligned}$$

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Now the simple observation

$$\int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) y \, d\mu(z) = \int_{\mathcal{F}_1^{\alpha_\ell}} f(\sigma_\ell z) y \, d\mu(z) + \int_{y=1}^{T} \int_{x=0}^{\alpha_\ell} f(\sigma_\ell z) y \, d\mu(z)$$

together with the identity

$$\begin{aligned}
&\mathrm{CT}_{s=0}\left[\int_{y=1}^{\infty} (A_\ell y^{s'} + B_\ell y^{1-s'}) \frac{dy}{y^{1+s}}\right] \\
&= \lim_{T\to\infty}\left[A_\ell\left(\int_1^T y^{s'} \frac{dy}{y} - \frac{T^{s'}}{s'}\right) + B_\ell\left(\int_1^T y^{1-s'} \frac{dy}{y} - \frac{T^{1-s'}}{1-s'}\right)\right] \\
&= \lim_{T\to\infty}\left[\int_{y=1}^{T}\int_{x=0}^{\alpha_\ell} f(\sigma_\ell z) y \, d\mu(z) - \left(A_\ell \frac{T^{s'}}{s'} + B_\ell \frac{T^{1-s'}}{1-s'}\right)\right]
\end{aligned}$$

gives the result when $s' \neq 0$. The $s' = 0$ case follows similary using the identity

$$\mathrm{CT}_{s=0}\left[\int_{y=1}^{\infty} \frac{dy}{y^{1+s}}\right] = \lim_{T\to\infty}\left[\int_1^T \frac{dy}{y} - \log(T)\right]. \qquad \square$$

5.4.1. The Laplacian. We now consider the action of the hyperbolic Laplacian on the regularized theta lift.

Theorem 5.9. *Let f be a weak Maass form for Γ with eigenvalue $\lambda = s'(1-s')$. Denote the constant term of the Fourier expansion of f at the cusp ℓ by $A_\ell y^{s'} + B_\ell y^{1-s'}$ with constants $A_\ell, B_\ell \in \mathbb{C}$. Then*

$$4\Delta_\tau I^{\mathrm{reg}}(\tau, f) = \begin{cases} \lambda I^{\mathrm{reg}}(\tau, f) & \text{if } s' \neq 0, 1, \\ -\displaystyle\sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \frac{A_\ell \varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau) & \text{if } s' = 0, \\ -\displaystyle\sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \frac{B_\ell \varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau) & \text{if } s' = 1. \end{cases}$$

To prove this theorem we need two propositions. We start by noting that in the view of Proposition 5.4, for $f \in A_0(\Gamma)$ which has at most linear exponential growth at the cusps of Γ, the integral

$$\int_M f(z)\Theta_L(\tau, z, \Delta_z\varphi_0) \, d\mu(z)$$

converges.

Proposition 5.10. *Let $f \in A_0(\Gamma)$ and assume that f has at most linear exponential growth at the cusps of Γ. Denote the constant term of the Fourier expansion of f at the cusp ℓ by $a_\ell(0, y)$. Then*

$$\begin{aligned}
\int_M f(z)\Theta_L(\tau, z, \Delta_z\varphi_0) \, d\mu(z) = \sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \lim_{T\to\infty}\Bigg[&\int_{\mathcal{F}_T^{\alpha_\ell}} (\Delta_z f)(\sigma_\ell z)\Theta_L(\tau, \sigma_\ell z, \varphi_0) \, d\mu(z) \\
&- \frac{\varepsilon_\ell \tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}}\left[\left(1 - y\frac{\partial}{\partial y}\right) a_\ell(0, y)\right]_{y=T}\Bigg].
\end{aligned}$$

In particular, the limit on the right hand side exists.

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Proof. According to (5.5), we have

$$\int_M f(z)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z) = \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \lim_{T\to\infty}\int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z)\Theta_L(\tau,\sigma_\ell z,\Delta_z\varphi_0)\,d\mu(z). \tag{5.10}$$

We use Stokes' theorem to rewrite the integral. For smooth functions f and g on D we have

$$\begin{aligned} -2i(\partial\bar\partial f)g &= (\Delta f)g\,d\mu(z) \\ &= f(\Delta g)\,d\mu(z) - 2i\,d\big((\bar\partial f)g + f(\partial g)\big). \end{aligned}\tag{5.11}$$

Consequently,

$$\begin{aligned} &\int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z)\Theta_L(\tau,\sigma_\ell z,\Delta_z\varphi_0)\,d\mu(z) \\ &\quad= \int_{\mathcal{F}_T^{\alpha_\ell}} (\Delta_z f)(\sigma_\ell z)\Theta_L(\tau,\sigma_\ell z,\varphi_0)\,d\mu(z) \\ &\qquad + 2i\int_{\partial\mathcal{F}_T^{\alpha_\ell}} \Big[\big(\bar\partial f(\sigma_\ell z)\big)\Theta_L(\tau,\sigma_\ell z,\varphi_0) + f(\sigma_\ell z)\big(\partial\Theta_L(\tau,\sigma_\ell z,\varphi_0)\big)\Big]. \end{aligned}$$

Using Proposition 5.4 and inserting the Fourier expansions of f at the cusps, we find for $T\to\infty$ that

$$\begin{aligned} &\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z)\Theta_L(\tau,\sigma_\ell z,\Delta_z\varphi_0)\,d\mu(z) \\ &\quad= \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \int_{\mathcal{F}_T^{\alpha_\ell}} (\Delta_z f)(\sigma_\ell z)\Theta_L(\tau,\sigma_\ell z,\varphi_0)\,d\mu(z) \\ &\qquad - 2i\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \frac{\tilde\Theta_{K_\ell}(\tau)}{\beta_\ell\sqrt{N}} \int_{z=0+iT}^{\alpha_\ell+iT} \left(\frac{\partial}{\partial\bar z}a_\ell(0,y)\right)y + a_\ell(0,y)\left(\frac{\partial}{\partial z}y\right)dx \\ &\qquad + O\left(\frac{1}{T}\right). \end{aligned}\tag{5.12}$$

The second term on the right hand side is equal to

$$-\sum_\ell \frac{\varepsilon_\ell\tilde\Theta_{K_\ell}(\tau)}{\sqrt{N}}\left[\left(1-y\frac{\partial}{\partial y}\right)a_\ell(0,y)\right]_{y=T}.$$

Inserting this into (5.12) and then into (5.10), we obtain the assertion. □

Proposition 5.11. *Let f be a weak Maass form of weight zero for Γ with eigenvalue $\lambda = s'(1-s')$. Denote the constant term of the Fourier expansion of f at the cusp ℓ by $a_\ell(0,y) = A_\ell y^{s'} + B_\ell y^{1-s'}$ with constants $A_\ell, B_\ell\in\mathbb{C}$. If $s'\neq 0,1$, then*

$$I^{\mathrm{reg}}(\tau,\Delta_z f) = \int_M f(z)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z).$$

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If $s' = 0$, then

$$0 = I^{\mathrm{reg}}(\tau, \Delta_z f) = \int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0)\, d\mu(z) + \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \frac{A_\ell \varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau).$$

If $s' = 1$, then

$$0 = I^{\mathrm{reg}}(\tau, \Delta_z f) = \int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0)\, d\mu(z) + \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \frac{B_\ell \varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau).$$

Proof. We use Proposition 5.10 for

$$\int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0)\, d\mu(z)$$

together with the fact that

$$\left[\left(1 - y\frac{\partial}{\partial y}\right) a_\ell(0, y)\right]_{y=T} = (1 - s')A_\ell T^{s'} + s' B_\ell T^{1-s'}.$$

Comparing this with the formula for $I^{\mathrm{reg}}(\tau, \Delta_z f)$ of Proposition 5.8, we obtain the assertion of the proposition. □

Proof of Theorem 5.9. According to Proposition 5.8 and Lemma 5.1 we have

$$\begin{aligned} 4\Delta_\tau I^{\mathrm{reg}}(\tau, f) &= 4 \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} f(\sigma_\ell z) \Delta_\tau \Theta_L(\tau, \sigma_\ell z, \varphi_0)\, d\mu(z) \\ &= \int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0). \end{aligned}$$

Now the assertion of the theorem follows from Proposition 5.11. □

5.5. Regularization using differential operators. Proposition 5.11 also leads to a different way of defining the regularized integral for weak Maass forms with non-zero eigenvalue. This regularization uses differential operators, in fact, the Laplace operator Δ_z on M. It is in the spirit of the regularized Siegel–Weil formula of Kudla–Rallis via regularized theta lifts [22]. For a related treatment, see Matthes [24].

Proposition 5.12. *Let f be a weak Maass form for Γ with eigenvalue $\lambda \neq 0$. Then*

$$I^{\mathrm{reg}}(\tau, f) = \frac{1}{\lambda} \int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0).$$

The integral on the right hand side converges absolutely.

Proof. According to Proposition 5.11, we have

$$\begin{aligned} I^{\mathrm{reg}}(\tau, f) &= \frac{1}{\lambda} I^{\mathrm{reg}}(\tau, \Delta_z f) \\ &= \frac{1}{\lambda} \int_M f(z)\Theta_L(\tau, z, \Delta_z \varphi_0). \end{aligned}$$

By Proposition 5.4, the integral converges. This proves the proposition. □

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The next lemma is a direct consequence of Proposition 5.12 and

$$\Delta_z \varphi_0(X,\tau,z) = 4\pi R_{-3/2}\varphi_1(X,\tau,z).$$

Lemma 5.13. *If f is a weak Maass form for Γ with eigenvalue $\lambda \neq 0$, then*

$$I^{\mathrm{reg}}(\tau,f) = \frac{4\pi}{\lambda} R_{-3/2}\Big(\int_M f(z)\Theta_L(\tau,z,\varphi_1)\,d\mu(z)\Big).$$

5.5.1. Eigenvalue zero. We now explain how the regularization by means of differential operators described in Proposition 5.12 can be extended to the case when the eigenvalue is 0. The idea is to use a 'spectral deformation' of f into a family of eigenfunctions.

Proposition 5.14. *Let $f \in H_0(\Gamma)$ be a harmonic weak Maass form. There exists an open neighborhood $U \subset \mathbb{C}$ of 1 and a holomorphic family of functions $(f_s)_{s\in U}$ on D such that $f_s(z)$ is a weak Maass form of weight 0 for Γ with eigenvalue $s(1-s)$, and $f_1 = f$.*

Proof. This result can be proved using Poincaré and Eisenstein series for Γ. See for example [12, Section 3], [16, p. 660], or [5, Proposition 1.12]. □

Let $f \in H_0(\Gamma)$ and $(f_s)_{s\in U}$ be as in Proposition 5.14. Denote the constant term of the Fourier expansion of f_s at the cusp ℓ by $a_\ell(0,y,s) = A_\ell(s)y^s + B_\ell(s)y^{1-s}$ with holomorphic functions $A_\ell(s)$, $B_\ell(s)$. In view of Proposition 5.12, we have for $s \in U \setminus \{1\}$ that

$$I^{\mathrm{reg}}(\tau,f_s) = \frac{1}{s(1-s)}\int_M f_s(z)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z).$$

The right hand side defines a meromorphic function for all $s \in U$. In view of Proposition 5.11, it has a first order pole at $s=1$ with residue

$$\sum_{\ell\in\Gamma\setminus\mathrm{Iso}(V)} \frac{B_\ell(1)\varepsilon_\ell}{\sqrt{N}}\tilde{\Theta}_{K_\ell}(\tau). \tag{5.13}$$

We can define a regularized theta integral by putting

$$J^{\mathrm{reg}}(\tau,f) := \mathrm{CT}_{s=1}\Big[\frac{1}{s(1-s)}\int_M f_s(z)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z)\Big]. \tag{5.14}$$

We now compare this with the regularized theta integral of Definition 5.6.

Proposition 5.15. *Let $f \in H_0(\Gamma)$ and $(f_s)_{s\in U}$ be as in Proposition 5.14. Denote the constant term of the Fourier expansion of f_s at the cusp ℓ by*

$$a_\ell(0,y,s) = A_\ell(s)y^s + B_\ell(s)y^{1-s}$$

with holomorphic functions $A_\ell(s)$, $B_\ell(s)$. Then we have

$$J^{\mathrm{reg}}(\tau,f) = I^{\mathrm{reg}}(\tau,f) + \sum_{\ell\in\Gamma\setminus\mathrm{Iso}(V)} \frac{B'_\ell(1)\varepsilon_\ell}{\sqrt{N}}\tilde{\Theta}_{K_\ell}(\tau).$$

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Proof. We put the Laurent expansion

$$f_s(z) = f(z) + f_1'(z)(s-1) + O((s-1)^2)$$

of f_s at $s = 1$ into the definition of $J^{\mathrm{reg}}(\tau, f)$. Here $f_1'(z)$ means the value of $f_s'(z) = \frac{\partial}{\partial s} f_s(z)$ at $s = 1$. Noticing that

$$\mathrm{CT}_{s=1}\left[\frac{1}{s(1-s)}\right] = 1,$$

we obtain that

$$J^{\mathrm{reg}}(\tau, f) = \int_M f(z)\Theta_L(\tau, z, \Delta_z\varphi_0)\, d\mu(z) - \int_M f_1'(z)\Theta_L(\tau, z, \Delta_z\varphi_0)\, d\mu(z). \tag{5.15}$$

According to Proposition 5.11, we get for the first term on the right hand side

$$\int_M f(z)\Theta_L(\tau, z, \Delta_z\varphi_0)\, d\mu(z) = -\sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \frac{B_\ell(1)\varepsilon_\ell}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau).$$

We compute the second term on the right hand side of (5.15) by means of Proposition 5.10. We have

$$\left[\left(1 - y\frac{\partial}{\partial y}\right) a_\ell(0, y, s)\right]_{y=T} = (1-s)A_\ell(s)T^s + sB_\ell(s)T^{1-s}.$$

Consequently, for the derivative with respect to s at $s = 1$ we find

$$\left[\left(1 - y\frac{\partial}{\partial y}\right) a_\ell'(0, y, 1)\right]_{y=T} = -A_\ell(1)T + B_\ell(1) + B_\ell'(1) - B_\ell(1)\log(T).$$

If we call this quantity $C_\ell(T)$, we obtain

$$\begin{aligned}\int_M & f_1'(z)\Theta_L(\tau, z, \Delta_z\varphi_0)\, d\mu(z)\\ &= \sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} \lim_{T\to\infty}\left[\int_{\mathcal{F}_T^{\alpha_\ell}} (\Delta_z f_1')(\sigma_\ell z)\Theta_L(\tau, \sigma_\ell z, \varphi_0) d\mu(z) - \frac{\varepsilon_\ell \tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}} C_\ell(T)\right].\end{aligned}$$

Since $\Delta f_s = s(1-s) f_s$, we have

$$\Delta f_1' = -f.$$

By means of Proposition 5.8, we get

$$\int_M f_1'(z)\Theta_L(\tau, z, \Delta_z\varphi_0)\, d\mu(z) = -I^{\mathrm{reg}}(\tau, f) - \sum_{\ell\in\Gamma\backslash \mathrm{Iso}(V)} (B_\ell(1) + B_\ell'(1))\frac{\varepsilon_\ell\tilde{\Theta}_{K_\ell}(\tau)}{\sqrt{N}}.$$

Inserting this into (5.15), we obtain the assertion. □

In particular, we see that the regularized theta integral $J^{\mathrm{reg}}(\tau, f)$ depends on the choice of the spectral deformation f_s. However, the dependency is mild, since only the derivatives of the constant terms in the Fourier expansions at $s = 1$ enter.

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6. The lift of Poincaré series and the regularized lift of j_m

6.1. Scalar-valued Poincaré series of weight 0. We construct scalar-valued Poincaré series of weight 0 for the group $\Gamma \subset G(\mathbb{Q})$. We will show in Section 6.3 that the theta lifts of these series are given by linear combinations of vector-valued Poincaré series constructed in Section 6.2. For simplicity we assume here that $\Gamma = \Gamma_0(N)$, since this is the only case which we will need later for the comparison of our results with [9]. We let

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

be the subgroup of translations.

Let $I_\nu(z)$ be the usual modified Bessel function as in [1, Chapter 9.6]. For $s \in \mathbb{C}$, $y \in \mathbb{R}_{>0}$ and $n \in \mathbb{Q}$, we let

$$\mathcal{I}_n(y,s) = \begin{cases} 2\pi |n|^{\frac{1}{2}} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n \neq 0, \\ y^s & \text{if } n = 0. \end{cases} \tag{6.1}$$

For $m \in \mathbb{Z}$ we define

$$G_m(z,s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} [\mathcal{I}_m(y,s) e(mx)]|_0 \gamma. \tag{6.2}$$

The series converges for $\mathrm{Re}(s) > 1$ and defines a weak Maass form of weight 0 for $\Gamma_0(N)$. It has the eigenvalue $s(1-s)$ under Δ_0. The function G_0 is the usual Eisenstein series while G_m for $m \neq 0$ was studied by Neunhöffer [25] and Niebur [26], among others. If $m \neq 0$, it follows from its Fourier expansion, Weil's bound and the properties of the I-Bessel function that $G_m(z,s)$ has a holomorphic continuation to $\mathrm{Re}(s) > \frac{3}{4}$. If $m < 0$, then $G_m(z,1) \in H_0^+(\Gamma)$.

6.2. Vector-valued Poincaré series of half-integral weight. We recall the definition of vector-valued Poincaré series for the Weil representation in a setup which is convenient for the present paper. These series are vector-valued analogues of the Poincaré series of weight $1/2$ considered in [9, Section 2].

Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittler functions (see [1, p. 190]). For $s \in \mathbb{C}$, $v \in \mathbb{R}_{>0}$ and $n \in \mathbb{Q}$, we let

$$\mathcal{M}_n(v,s) = \begin{cases} \Gamma(2s)^{-1}(4\pi|n|v)^{-1/4} M_{\frac{1}{4}\operatorname{sgn} n,\, s-\frac{1}{2}}(4\pi|n|v) & \text{if } n \neq 0, \\ v^{s-\frac{1}{4}} & \text{if } n = 0. \end{cases} \tag{6.3}$$

For $h \in L'/L$, and $m \in \mathbb{Z} + Q(h)$ we define

$$P_{m,h}(\tau,s) = \frac{1}{2} \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'} [\mathcal{M}_m(v,s) e(mu) \mathfrak{e}_h]|_{1/2,L}\gamma. \tag{6.4}$$

Here we have set $\Gamma'_\infty := \langle T \rangle \subset \tilde{\Gamma}$ with $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$. The series converges for $\mathrm{Re}(s) > 1$ and defines a weak Maass form of weight $1/2$ for Γ' with representation ρ_L. It has the eigenvalue $(s - \frac{1}{4})(\frac{3}{4} - s)$ under $\Delta_{1/2}$. When $Q(h) \in \mathbb{Z}$ and $m = 0$, the function $P_{0,h}(\tau,s)$ is a vector-valued Eisenstein series of weight $1/2$.

For L as in Example 2.1 we may apply the map (2.3) to $P_{m,h}(\tau,s)$ for $m = \frac{d}{4N}$ and $d \in \mathbb{Z}$ to obtain the scalar-valued Poincaré series $P_d^+(\tau,s)$ considered in [9].

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6.3. The lift of Poincaré series. We now assume that L is the lattice defined in Example 2.1. We identify the finite quadratic module L'/L with $\mathbb{Z}/2N\mathbb{Z}$ equipped with the quadratic form $r \mapsto r^2/4N$. Moreover, we assume that $\Gamma = \Gamma_0(N)$. In this section we will explicitly calculate the regularized lift of the Poincaré series $G_{-m}(z,s)$ defined in Section 6.1 for $m \geq 0$.

For the cusp $\ell_0 = \infty$, we can realize the space $W = V \cap \ell_0^{\perp}/\ell_0$ as $\mathbb{Q}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$. Hence $K := K_{\ell_0} = \mathbb{Z}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$. Moreover we have $L'/L \simeq K'/K$. For $\alpha, \beta \in W(\mathbb{R})$ we define the $\mathbb{C}[K'/K]$-valued theta series

$$\Theta_K(\tau,\alpha,\beta) = \sum_{\lambda \in K'} e\big(Q(\lambda+\beta)\tau - (\lambda+\beta/2,\alpha)\big)\mathfrak{e}_{\lambda+K},$$

and we write $\theta_{K,h}(\tau,\alpha,\beta)$ for the individual components. Notice that the theta function $\Theta_K(\tau)$ defined earlier is equal to $\Theta_K(\tau,0,0)$. The following proposition is a special case of [4, Theorem 5.2].

Proposition 6.1. *We have the identity*

$$\Theta_L(\tau,z,\varphi_0) = \sqrt{N}y\Theta_K(\tau,0,0) + \frac{\sqrt{N}y}{2}\sum_{n=1}^{\infty}\sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'}\left[\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\Theta_K(\tau,nx,0)\right]\bigg|_{1/2,K}\gamma.$$

Theorem 6.2. *Assume that* $\mathrm{Re}(s) > 1$. *If m is a positive integer, then*

$$I^{\mathrm{reg}}(\tau, G_{-m}(z,s)) = \sqrt{\pi N}\,\Gamma\left(\frac{s}{2}\right)\sum_{n|m} P_{\frac{m^2}{4Nn^2},\frac{m}{n}}\left(\tau,\frac{s}{2}+\frac{1}{4}\right).$$

For $m = 0$, then

$$I^{\mathrm{reg}}(\tau, G_0(z,s)) = \frac{N^{\frac{1}{2}-\frac{s}{2}}}{2}\zeta^*(s)P_{0,0}\left(\tau,\frac{s}{2}+\frac{1}{4}\right).$$

Here

$$\zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and $P_{m,h}(\tau,s)$ denotes the $\mathbb{C}[L'/L]$-valued weight $1/2$ Poincaré series defined in Section 6.2.

Proof. According to Proposition 5.12 we have

$$I^{\mathrm{reg}}(\tau, G_{-m}(z,s)) = \frac{1}{s(1-s)}\int_M G_{-m}(z,s)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z).$$

The theta function on the right hand side is square exponentially decreasing at all cusps. Hence, by the usual unfolding argument, we find that

$$I^{\mathrm{reg}}(\tau, G_{-m}(z,s)) = \frac{1}{s(1-s)}\int_{\Gamma_\infty\backslash\mathbb{H}} \mathcal{I}_{-m}(y,s)e(-mx)\Theta_L(\tau,z,\Delta_z\varphi_0)\,d\mu(z).$$

By Proposition 6.1, we may replace $\Theta_L(\tau,z,\Delta_{0,z}\varphi_0)$ by $\Delta_{0,z}\tilde{\Theta}_L(\tau,z,\varphi_0)$, where

$$\tilde{\Theta}_L(\tau,z,\varphi_0) = \frac{\sqrt{N}y}{2}\sum_{n=1}^{\infty}\sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'}\left[\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\Theta_K(\tau,nx,0)\right]\bigg|_{1/2,K}\gamma.$$

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The function $\tilde{\Theta}_L(\tau, z, \varphi_0)$ and its partial derivatives have square exponential decay as $y \to \infty$. Therefore, for $\mathrm{Re}(s)$ large, we may move the Laplace operator to $\mathcal{I}_{-m}(y, s)e(-mx)$ to obtain

$$(6.5)\quad I^{\mathrm{reg}}(\tau, G_{-m}(z, s)) = \frac{1}{s(1-s)} \int_{\Gamma_\infty \backslash \mathbb{H}} \big(\Delta_z \mathcal{I}_{-m}(y, s) e(-mx)\big) \tilde{\Theta}_L(\tau, z, \varphi_0)\, d\mu(z)$$
$$= \int_{\Gamma_\infty \backslash \mathbb{H}} \mathcal{I}_{-m}(y, s) e(-mx) \tilde{\Theta}_L(\tau, z, \varphi_0)\, d\mu(z)$$
$$= \frac{\sqrt{N}}{2} \sum_{n=1}^{\infty} \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'} I(\tau, s, m, n)|_{1/2,K}\gamma,$$

where

$$I(\tau, s, m, n) = \int_{y=0}^{\infty} \int_{x=0}^{1} \mathcal{I}_{-m}(y, s) e(-mx) \exp\left(-\frac{\pi N n^2 y^2}{v}\right) \Theta_K(\tau, nx, 0)\, y\, d\mu(z).$$

Using the fact that $K' = \mathbb{Z}\begin{pmatrix} 1/2N & \\ & -1/2N \end{pmatrix}$ and the identification $K'/K \cong \mathbb{Z}/2N\mathbb{Z}$, we have

$$\Theta_K(\tau, nx, 0) = \sum_{b \in \mathbb{Z}} e\left(\frac{b^2}{4N}\tau - nbx\right) \mathfrak{e}_b.$$

Inserting this in the formula for $I(\tau, s, m, n)$, and by integrating over x, we see that $I(\tau, s, m, n)$ vanishes when $n \nmid m$. If $n \mid m$, then only the summand for $b = -m/n$ occurs, and so

$$(6.6)\qquad I(\tau, s, m, n) = \int_0^\infty \mathcal{I}_{-m}(y, s) \exp\left(-\frac{\pi N n^2 y^2}{v}\right) \frac{dy}{y} e\left(\frac{m^2}{4Nn^2}\tau\right) \mathfrak{e}_{-m/n}.$$

We first compute the latter integral for $m > 0$. In this case we have

$$\mathcal{I}_{-m}(y, s) = 2\pi m^{\frac{1}{2}} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi m y).$$

Inserting this and substituting $t = y^2$ in the integral, we obtain

$$\int_0^\infty \mathcal{I}_{-m}(y, s) \exp\left(-\frac{\pi N n^2 y^2}{v}\right) \frac{dy}{y}$$
$$= 2\pi \int_0^\infty \sqrt{my}\, I_{s-1/2}(2\pi m y) \exp\left(-\frac{\pi N n^2 y^2}{v}\right) \frac{dy}{y}$$
$$= \pi\sqrt{m} \int_0^\infty I_{s-1/2}(2\pi m \sqrt{t}) \exp\left(-\frac{\pi N n^2 t}{v}\right) t^{-3/4}\, dt.$$

The latter integral is a Laplace transform which is computed in [11, equation (20) on p. 197]. Inserting the evaluation, we obtain

$$\int_0^\infty \mathcal{I}_{-m}(y, s) \exp\left(-\frac{\pi N n^2 y^2}{v}\right) \frac{dy}{y}$$
$$= \frac{\sqrt{\pi}\,\Gamma(s/2)}{\Gamma(s+1/2)} \left(\frac{Nn^2}{\pi m^2 v}\right)^{1/4} M_{1/4, s/2-1/4}\left(\frac{\pi m^2 v}{Nn^2}\right) \exp\left(\frac{\pi m^2 v}{2Nn^2}\right)$$
$$= \sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right) \mathcal{M}_{\frac{m^2}{4Nn^2}}\left(v, \frac{s}{2} + \frac{1}{4}\right) \exp\left(\frac{\pi m^2 v}{2Nn^2}\right).$$

Consequently, we have in the case $n \mid m$ that

$$I(\tau, s, m, n) = \sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right) \mathcal{M}_{\frac{m^2}{4Nn^2}}\left(v, \frac{s}{2} + \frac{1}{4}\right) e\left(\frac{m^2}{4Nn^2}u\right) \mathfrak{e}_{-m/n}.$$

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Substituting this in (6.5), we see that

$$I^{\mathrm{reg}}(\tau, G_{-m}(z,s)) = \sqrt{\pi N}\,\Gamma\left(\frac{s}{2}\right)\sum_{n\mid m} P_{\frac{m^2}{4Nn^2},-\frac{m}{n}}\left(\tau, \frac{s}{2}+\frac{1}{4}\right).$$

Since $P_{m,h}(\tau,s) = P_{m,-h}(\tau,s)$, this concludes the proof of the theorem for $m > 0$.

We now compute integral in (6.6) for $m = 0$. In this case we have $\mathcal{I}_0(y,s) = y^s$. Inserting this into (6.6), we find

$$\begin{aligned}\int_0^\infty \mathcal{I}_0(y,s)\exp\left(-\frac{\pi N n^2 y^2}{v}\right)\frac{dy}{y} &= \int_0^\infty \exp\left(-\frac{\pi N n^2 y^2}{v}\right) y^{s-1}\,dy\\ &= \frac{\Gamma(s/2)}{2}\left(\frac{v}{\pi N n^2}\right)^{s/2}.\end{aligned}$$

Hence, we obtain

$$I(\tau,s,m,n) = \frac{\Gamma(\frac{s}{2})}{2}\left(\frac{v}{\pi N n^2}\right)^{s/2}\mathfrak{e}_0.$$

Substituting into (6.5), we see that

$$\begin{aligned}I^{\mathrm{reg}}(\tau, G_0(z,s)) &= \frac{N^{\frac{1}{2}-\frac{s}{2}}}{4}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'} v^{s/2}\mathfrak{e}_0|_{1/2,K}\gamma\\ &= \frac{N^{\frac{1}{2}-\frac{s}{2}}}{2}\zeta^*(s)P_{0,0}\left(\tau,\frac{s}{2}+\frac{1}{4}\right).\end{aligned}$$

This concludes the proof of the theorem for $m = 0$. □

6.4. The case of level 1. As an application, we consider the special case $N = 1$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We compute the lift of the space $M_0^!(\Gamma) = \mathbb{C}[j]$. A basis for this space is given by the functions j_m for $m \in \mathbb{Z}_{\geq 0}$ whose Fourier expansion starts as

$$j_m(z) = q^{-m} + O(q).$$

For instance, $j_0 = 1$ and $j_1 = j - 744$.

We begin by computing the lift of the constant function in terms of Eisenstein series. As a spectral deformation of the constant function $j_0 = 1$ in the sense of Proposition 5.14 we choose

$$j_0(z,s) = \frac{\zeta^*(2)}{\zeta^*(2s-1)}G_0(z,s).$$

It is well known that $G_0(z,s)$ has a first order pole at $s = 1$, which cancels out against the pole of $\zeta^*(2s-1)$. We have $j_0(z,1) = 1$, and the constant term of $j_0(z,s)$ at the cusp $\ell_0 = \infty$ is given by $A_\infty(s)y^s + B_\infty(s)y^{1-s}$ with

$$\begin{aligned}A_\infty(s) &= \frac{\zeta^*(2)}{\zeta^*(2s-1)} = \frac{\pi}{3}(s-1) + O((s-1)^2),\\ B_\infty(s) &= \frac{\zeta^*(2)}{\zeta^*(2s)} = 1 + \left(\gamma + \log(\pi) - \frac{12\zeta'(2)}{\pi^2}\right)(s-1) + O((s-1)^2).\end{aligned}$$

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According to Theorem 6.2, for $\mathrm{Re}(s) > 1$, the lift of $j_0(z,s)$ is equal to

$$I^{\mathrm{reg}}(\tau, j_0(z,s)) = \frac{\pi\zeta^*(s)}{12\zeta^*(2s-1)} P_{0,0}\left(\tau, \frac{s}{2} + \frac{1}{4}\right). \tag{6.7}$$

By (5.13), the right hand side has a meromorphic continuation to $\mathbb{C}$ with a first order pole at $s = 1$ with residue $B_\infty(1)\Theta_K(\tau) = \Theta_K(\tau)$. In particular, we see that $P_{0,0}(\tau, \frac{s}{2} + \frac{1}{4})$ has a first order pole at $s = 1$ with residue $\frac{6}{\pi}\Theta_K(\tau)$. We obtain the following corollary to Theorem 6.2.

Corollary 6.3. *We have*

$$J^{\mathrm{reg}}(\tau, 1) = \mathrm{CT}_{s=1}\left[\frac{\pi\zeta^*(s)}{12\zeta^*(2s-1)} P_{0,0}\left(\tau, \frac{s}{2} + \frac{1}{4}\right)\right],$$
$$I^{\mathrm{reg}}(\tau, 1) = J^{\mathrm{reg}}(\tau, 1) - B'_\infty(1)\Theta_K(\tau).$$

We now compute the lift of j_m for $m > 0$ in terms of Poincaré series. It follows from the Fourier expansion, Weil's bound and the properties of the I-Bessel function that $G_{-m}(z,s)$ has a holomorphic continuation to $\mathrm{Re}(s) > \frac{3}{4}$. The constant term of the Fourier expansion of $G_{-m}(z,s)$ is equal to

$$\frac{4\pi m^{1-s}\sigma_{2s-1}(m)}{(2s-1)\zeta^*(2s)} y^{1-s},$$

where $\sigma_{2s-1}(m) = \sum_{d|m} d^{2s-1}$, see e.g. [12, 26]. We define

$$j_m(z,s) := G_{-m}(z,s) - \frac{4\pi m^{1-s}\sigma_{2s-1}(m)}{(2s-1)\zeta^*(2s-1)} G_0(z,s). \tag{6.8}$$

This function has an analytic continuation to $\mathrm{Re}(s) > 3/4$. The constant term in its Fourier expansion is given by $A_\infty(s)y^s + B_\infty(s)y^{1-s}$ with

$$A_\infty(s) = -\frac{4\pi m^{1-s}\sigma_{2s-1}(m)}{(2s-1)\zeta^*(2s-1)},$$
$$B_\infty(s) = 0.$$

Moreover, we have

$$j_m(z,1) = j_m(z). \tag{6.9}$$

Hence, we may use the functions $j_m(z,s)$ as spectral deformations of the $j_m(z)$.

According to Theorem 6.2, for $\mathrm{Re}(s) > 1$, the lift of $j_m(z,s)$ is equal to

$$I^{\mathrm{reg}}(\tau, j_m(\cdot,s)) = \sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\sum_{n|m} P_{\frac{m^2}{4n^2},\frac{m}{n}}\left(\tau, \frac{s}{2} + \frac{1}{4}\right) + A_\infty(s)\frac{\zeta^*(s)}{2} P_{0,0}(\tau, \frac{s}{2} + \frac{1}{4}).$$

By (5.13), the right hand side has a holomorphic continuation to a neighborhood of $s = 1$. Its value at $s = 1$ is equal to the regularized integral $J^{\mathrm{reg}}(\tau, j_m)$. Since $B_\infty(s) = 0$, we obtain the following corollary to Proposition 5.15.

Corollary 6.4. *Assume $m > 0$. Then*

$$I^{\mathrm{reg}}(\tau, j_m) = J^{\mathrm{reg}}(\tau, j_m).$$

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7. A Green function for φ_0

In this section, we introduce a Green function η for the Schwartz function φ_0. Its properties will be the key for the proof of the results in Section 4.

7.1. The singular function η. We first recall the definition of Kudla's Green function ξ for φ_1 (see [18, Section 11] and for our setting Remark 5.3). It is defined for non-zero vectors $X \in V(\mathbb{R})$ and given by

$$\xi(X,\tau,z) = v^{3/2} E_1(2\pi v R(X,z)) e(Q(X)\bar{\tau}) = v^{3/2}\left(\int_1^\infty e^{-2\pi v R(X,z)t}\frac{dt}{t}\right) e(Q(X)\bar{\tau}). \tag{7.1}$$

Here

$$E_1(w) = \int_w^\infty e^{-t}\frac{dt}{t}$$

with $w \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ is the exponential integral as in [1]. Since

$$E_1(w) = -\gamma - \log(w) + \int_0^w (1-e^{-t})\frac{dt}{t}, \tag{7.2}$$

(the last function on the right hand is entire and is denoted by $\mathrm{Ein}(w)$) we directly see that ξ has a logarithmic singularity for $z = D_X$ when $R(X,z) = 0$ and is smooth for $Q(X) \geq 0$. Outside the singularity one has, see [18],

$$dd^c \xi(X,\tau,z) = \varphi_1(X,\tau,z)\, d\mu(z), \tag{7.3}$$

which can be also obtained via Lemma 5.2. Here $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ so that

$$dd^c = -\frac{1}{2\pi i}\partial\bar{\partial} = -\frac{1}{4\pi}\Delta_z d\mu(z).$$

For the relationship between ξ and φ_1 as currents, see (7.6).

We now define for $X \neq 0$ our Green function η by

$$\eta(X,\tau,z) = \pi\left(\int_v^\infty E_1(2\pi R(X,z)t)e^{2\pi(X,X)t}\frac{dt}{\sqrt{t}}\right) e(Q(X)\tau).$$

We often drop the dependence on τ in the notation. We easily calculate

$$\partial_z \eta(X,z) = -\pi \operatorname{sgn}(X,X(z))\frac{(X,X'(z))}{R(X,z)} \operatorname{erfc}\left(\sqrt{\pi v}|(X,X(z))|\right) e(Q(X)\tau)\, dz, \tag{7.4}$$

where $X'(z) = \frac{\partial}{\partial z}X(z)$ and

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-r^2}\, dr$$

is the complimentary error function. Note that

$$X'(z) = \frac{i}{2y}X(z) + \frac{1}{\sqrt{N}y}\begin{pmatrix} -\frac{1}{2} & \bar{z} \\ 0 & \frac{1}{2}\end{pmatrix}.$$

For $Q(X) \neq 0$, we now analyze the singularities of η in more detail.

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Lemma 7.1. (i) *Let $X \in V(\mathbb{R})$ such that $Q(X) = m < 0$. Then η has a logarithmic singularity at $z = z_X$. More precisely,*

$$\tilde{\eta}(X,z) := \eta(X,z) + \pi \frac{\operatorname{erfc}(2\sqrt{\pi|m|v})}{\sqrt{|m|}} e(m\tau) \log|z - z_X|^2$$

is a smooth function in a neighborhood of z_X. Furthermore, $\eta(X)$ and its derivatives $\partial\eta(X)$, $\bar{\partial}\eta(X)$ are square exponential decreasing (in the coordinates x, y of z) at the boundary of D.

(ii) *Let $X \in V(\mathbb{R})$ such that $Q(X) = m > 0$. Then $\eta(X,z)$ is differentiable, but* not *C^1. The 1-form $\partial\eta(X,z)$ is discontinuous at the cycle $c_X = \{z \in D : (X, X(z)) = 0\}$, and outside the cycle c_X we have*

$$\partial\eta(X,z) = \frac{\pi i}{2} \operatorname{sgn}(X, X(z)) \operatorname{erfc}\big(\sqrt{\pi v}|(X, X(z))|\big) e(m\tau)\, dz_X.$$

Furthermore, assume that $\overline{\Gamma}_X$ is infinitely cyclic. Then $\eta(X)$ and its derivatives $\partial\eta(X)$, $\bar{\partial}\eta(X)$ are square exponential decreasing (in the coordinates x, y) at the boundary of the 'tube' $\Gamma_X \backslash D$.

Proof. For (i), we have $m < 0$. Via (7.2) we therefore immediately see that

$$\eta(X,z) + \pi \log R(X,z) \left(\int_v^\infty e^{-4\pi|m|t} \frac{dt}{\sqrt{t}} \right) e(m\tau)$$

is smooth. By (3.2) the singularity of $\eta(X,z)$ at $z = z_X$ is hence given by

$$-\pi \left(\int_v^\infty e^{-4\pi|m|t} \frac{dt}{\sqrt{t}} \right) e(m\tau) \log|z - z_X|^2 = -\pi \frac{\operatorname{erfc}(2\sqrt{\pi|m|v})}{\sqrt{|m|}} e(m\tau) \log|z - z_X|^2.$$

Since $E_1(w) \leq e^{-w}/w$, we have

$$|\eta(X)| \leq \pi \left(\int_v^\infty \frac{e^{-\pi(X,X(z))^2}}{R(X,z)} t^{-3/2}\, dt \right) e(m\tau).$$

Now the growth behavior follows from

$$(X, X(z))^2 = \frac{N}{y^2} \big(x_3|z|^2 - 2x_1 \operatorname{Re}(z) - x_2 \big)^2$$

for $X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & -x_2 \end{smallmatrix}\right)$. Note that since $Q(X) < 0$, we must have $x_3 \neq 0$.

By the G-equivariance properties of η, $X(z)$, and dz_X, it suffices to show (ii) for

$$X = \pm \begin{pmatrix} \sqrt{m/N} & \\ & -\sqrt{m/N} \end{pmatrix}.$$

Then we have $(X, X(z)) = \mp 2x\sqrt{m}/y$, $R(X,z) = 2m|z|^2/y^2$, $(X, \partial X(z)) = i\sqrt{m}\bar{z}/y^2\, dz$, and $dz_X = \pm dz/\sqrt{m}z$. Hence by (7.4) we obtain

$$\partial\eta(X) = -\operatorname{sgn}(x) \frac{\pi i}{2\sqrt{m}} \operatorname{erfc}\left(2\sqrt{\pi v m} \frac{|x|}{y} \right) e(m\tau) \frac{dz}{z}, \tag{7.5}$$

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which is the asserted equality for this X. For this X, we have $\overline{\Gamma}_X = \langle\left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1}\end{smallmatrix}\right)\rangle$ with some $r > 1$. Hence c_X is the imaginary axis and a fundamental domain for $\Gamma_X \backslash D$ is given by the annulus $\{z \in D : 1 \leq |z| < r\}$. Then (7.5) implies the very rapid decay in $\Gamma_X \backslash D$. □

We now state the analog to (7.3).

Proposition 7.2. *Outside the singularities,*

$$dd^c \eta(X, \tau, z) = \varphi_0(X, \tau, z)\, d\mu(z).$$

Proof. Using (7.3) we compute

$$\begin{aligned}
dd^c \eta(X, \tau, z) &= \pi \left(\int_v^\infty dd^c E_1(2\pi R(X, z)t) e^{2\pi(X,X)t} \frac{dt}{\sqrt{t}} \right) e^{\pi i (X,X)\tau} \\
&= \pi \left(\int_v^\infty t^{-3/2} \varphi_1(X, u + it, z) e^{-\pi i (X,X)(u+it)} \frac{dt}{\sqrt{t}} \right) e^{\pi i (X,X)\tau}\, d\mu(z) \\
&= - \left(\int_v^\infty t^{-2} \big(L_{\frac{1}{2}} \varphi_0(X, u + it, z)\big) e^{-\pi i (X,X)(u+it)}\, dt \right) e^{\pi i (X,X)\tau}\, d\mu(z) \\
&= - \left(\int_v^\infty \frac{\partial}{\partial t} \big[\sqrt{t} e^{-\pi (X, X(z))^2 t}\big]\, dt \right) e^{\pi i (X,X)\tau}\, d\mu(z) \\
&= \varphi_0(X, \tau, z)\, d\mu(z).
\end{aligned}$$

Here we used $\varphi_0(X, u + it, z) = \sqrt{t} e^{-\pi (X, X(z))^2 t} e^{\pi i (X,X)(u+it)}$. □

The next lemma gives the relationship to Kudla's Green function.

Lemma 7.3. *Outside the singularity D_X,*

$$L_{\frac{1}{2}} \eta(X, \tau, z) = -\pi \xi(X, \tau, z).$$

Proof. We compute

$$\begin{aligned}
L_{\frac{1}{2}} \eta(X, \tau, z) &= -2\pi i v^2 \frac{\partial}{\partial \bar{\tau}} \left(\int_v^\infty E_1(2\pi R(X, z)t) e^{2\pi(X,X)t} \frac{dt}{\sqrt{t}} \right) e(Q(X)\tau) \\
&= \pi v^2 \left(\frac{\partial}{\partial v} \int_v^\infty E_1(2\pi R(X, z)t) e^{2\pi(X,X)t} \frac{dt}{\sqrt{t}} \right) e(Q(X)\tau) \\
&= -\pi \xi(X, \tau, z),
\end{aligned}$$

as claimed. □

To summarize, we have the following diagram:

$$\begin{array}{ccc}
\eta(X, \tau, z) & \xmapsto{\;-\frac{1}{\pi} L_{1/2}\;} & \xi(X, \tau, z) \\
\Big\downarrow{\scriptstyle dd^c} & & \Big\downarrow{\scriptstyle dd^c} \\
\varphi_0(X, \tau, z) d\mu(z) & \xmapsto{\;-\frac{1}{\pi} L_{1/2}\;} & \varphi_1(X, \tau, z)\, d\mu(z).
\end{array}$$

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7.2. Current equations. We now consider η as a current. The current equations we obtain for η can be viewed as a refinement of Kudla's current equation for ξ, see [18, Proposition 11.1], namely, for $X \neq 0$,

$$dd^c[\xi(X,\tau,z)] + v^{3/2}e(m\bar{\tau})\delta_{D_X} = [\varphi_1(X,\tau,z)\,d\mu(z)], \tag{7.6}$$

as currents acting on functions with compact support on D. Here $D_X = \emptyset$ if $Q(X) \geq 0$. We recover (7.6) by applying the lowering operator $L_{1/2}$ to the current equations for η below.

We first note that by Proposition 7.2 for a C^2-function f on D we have

$$\begin{aligned} 2\pi i f(z)\varphi_0(X,z)d\mu(z) = d\big(f(z)\partial\eta(X,z)\big) - d\big(\bar{\partial}f(z)\eta(X,z)\big) \\ + \partial\bar{\partial}f(z)\eta(X,z), \end{aligned} \tag{7.7}$$

away from the singularities of η.

7.2.1. The elliptic case. Throughout this subsection we assume that $X \in V$ is a vector of length $Q(X) = m < 0$. Then the stabilizer $\overline{\Gamma}_X$ of X is finite.

Proposition 7.4. *The function $\eta(X,\tau,z)$ satisfies the following current equation:*

$$dd^c[\eta(X,\tau)] + \pi\frac{\operatorname{erfc}(2\sqrt{\pi|m|v})}{2\sqrt{|m|}}e(m\tau)\delta_{D_X} = [\varphi_0(X,\tau)\,d\mu(z)]$$

as currents on C^2-functions on D with at most linear exponential growth, that is, for such f,

$$\begin{aligned} \int_D f(z)\varphi_0(X,\tau,z)\,d\mu(z) = \frac{\pi e^{2\pi m\tau}}{2\sqrt{|m|}}\operatorname{erfc}(2\sqrt{\pi|m|v})f(z_X) \\ - \frac{1}{4\pi}\int_D(\Delta f(z))\eta(X,\tau,z)\,d\mu(z). \end{aligned}$$

Proof. For functions with compact support this can be easily seen using (7.7), Stokes' theorem, and the logarithmic singularity of η. In fact, it is very special case of the Poincaré–Lelong Lemma, see e.g. [29, pp. 41–42]. For functions with at most linear exponential growth the same argument goes through since $\eta(X)$ and its derivatives are square exponentially decreasing. □

Corollary 7.5. *Let $f \in H_0^+(\Gamma)$. Then*

$$\int_M f(z)\sum_{\gamma\in\Gamma_X\backslash\Gamma}\varphi_0(X,\tau,\gamma z)\,d\mu(z)$$

converges, and

$$\begin{aligned} \int_M f(z)\sum_{\gamma\in\Gamma_X\backslash\Gamma}\varphi_0(X,\tau,\gamma z)\,d\mu(z) &= \frac{1}{|\overline{\Gamma}_X|}\int_D f(z)\varphi_0(X,\tau,z)\,d\mu(z) \\ &= \frac{\pi e^{2\pi m\tau}}{2\sqrt{|m|}}\operatorname{erfc}(2\sqrt{\pi|m|v})\frac{1}{|\overline{\Gamma}_X|}f(D_X). \end{aligned}$$

Proof. This is immediate from Proposition 7.4 and the linear exponential growth of weak Maass forms. □

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7.2.2. The non-split hyperbolic case. Throughout this subsection we will assume that $X \in V$ is a vector of positive length $Q(X) = m > 0$. In addition, we assume that the stabilizer $\overline{\Gamma}_X$ is infinite cyclic.

Proposition 7.6. *For X as above, the function $\eta(X, \tau, z)$ satisfies the following current equation:*

$$dd^c[\eta(X,\tau)] + \frac{1}{2}e(m\tau)\delta_{c(X),dz_X} = [\varphi_0(X,\tau)\, d\mu(z)]$$

as currents on C^2-functions on $\Gamma_X \backslash D$ with at most linear exponential growth. That is, for such f,

$$\int_{\Gamma_X\backslash D} f(z)\varphi_0(X,\tau,z)\, d\mu(z) = \frac{1}{2}e(m\tau)\int_{c(X)} f(z)\, dz_X - \frac{1}{4\pi}\int_{\Gamma_X\backslash D} (\Delta f(z))\eta(X,\tau,z)\, d\mu(z).$$

Proof. We can assume that $X = \sqrt{m/N}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$ so that

$$c_X = \left\{ z \in D : (X, X(z)) = -\frac{x}{y} = 0 \right\}$$

is the imaginary axis, $\overline{\Gamma}_X = \left\langle \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1}\end{smallmatrix}\right)\right\rangle$, and $\Gamma_X \backslash c_X$ inside the annulus

$$\Gamma_X \backslash D = \{z \in D : 1 \le |z| \le r\}$$

is given by $\{z = iy : 1 \le y \le r\}$. We define an ε-neighborhood for c_X in $\Gamma_X \backslash D$ by

$$U_\varepsilon(c_X) = \left\{ z \in \Gamma_X\backslash D : |(X, X(z))| = \frac{|x|}{y} < \varepsilon \right\}.$$

We have

$$\int_{\Gamma_X\backslash D} f(z)\varphi_0(X,\tau,z)\, d\mu(z) = \lim_{\varepsilon\to 0}\int_{\Gamma_X\backslash D - U_\varepsilon(c_X)} f(z)\varphi_0(X,\tau,z)\, d\mu(z).$$

Using (7.7) we see that for fixed ε the integral on the right hand side is equal to

$$-\frac{1}{2\pi i}\int_{\partial U_\varepsilon(c_X)} f(z)\partial\eta(X,z) + \frac{1}{2\pi i}\int_{\partial U_\varepsilon(c_X)} \bar\partial f(z)\eta(X,z) + \frac{1}{2\pi i}\int_{\Gamma_X\backslash D - U_\varepsilon(c_X)} \partial\bar\partial f(z)\eta(X,z).$$

(Note $\partial U_\varepsilon(c_X) = -\partial(\Gamma_X\backslash D - U_\varepsilon(c_X))$.) Here we also used the very rapid decay of φ_0, η, and $\partial\eta$ at the boundary of the tube $\Gamma_X\backslash D$. As $\varepsilon \to 0$ the last term becomes

$$\int_{\Gamma_X\backslash D} (dd^c f(z))\eta(X,\tau,z)\, d\mu(z).$$

The second term vanishes since η is continuous. For the first term, we define

$$c_{X,\pm\varepsilon} = \left\{ z \in \Gamma_X\backslash D : -(X, X(z)) = \frac{x}{y} = \pm\varepsilon \right\}.$$

Then by (7.5) we obtain

$$\int_{\partial U_\varepsilon(c_X)} f(z)\partial\eta(X,z) = \left[\int_{c_{X,\varepsilon}} f(z)\, dz_X + \int_{c_{X,-\varepsilon}} f(z)\, dz_X\right] \operatorname{erfc}\!\left(\sqrt{\pi m v}\varepsilon\right) e(m\tau).$$

Taking the limit completes the proof. □

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This holds in particular if f is a weak Maass form of weight 0. As an immediate consequence of Proposition 7.6, we obtain the following result.

Corollary 7.7. *Let $f \in H_0^+(\Gamma)$. Then*

$$\int_M f(z) \sum_{\gamma \in \Gamma_X \backslash \Gamma} \varphi_0(X, \tau, \gamma z)\, d\mu(z)$$

converges, and

$$\begin{aligned}\int_M f(z) \sum_{\gamma \in \Gamma_X \backslash \Gamma} \varphi_0(X, \tau, \gamma z)\, d\mu(z) &= \int_{\Gamma_X \backslash D} f(z) \varphi_0(X, \tau, z)\, d\mu(z) \\ &= \frac{1}{2} e(m\tau) \int_{c(X)} f(z)\, dz_X.\end{aligned}$$

Remark 7.8 (The split hyperbolic case). Assume that $X \in V$ is a vector of positive length $Q(X) = m > 0$ such that the stabilizer $\overline{\Gamma}_X$ is trivial. Hence $\Gamma_X \backslash D = D$. Then Proposition 7.6 carries over to the present situation if one assumes that f is a function of compact support on D. However, for a function f not of sufficient decay,

$$\int_M f(z) \sum_{\gamma \in \overline{\Gamma}} \varphi_0(X, \tau, \gamma z)\, d\mu(z)$$

does *not* converge. In fact, exactly these terms require the theta lift to be regularized.

8. The Fourier expansion of the regularized theta lift

In this section, we give the proofs for the results stated in Section 4. We set

$$\theta_{m,h}(\tau, z) = \sum_{X \in L_{m,h}} \varphi_0(X, \tau, z), \tag{8.1}$$

which defines a Γ-invariant function on D. We then have

$$I_h(\tau, f) = \sum_{m \in \mathbb{Q}} \int_M^{\mathrm{reg}} f(z) \theta_{m,h}(\tau, z)\, d\mu(z), \tag{8.2}$$

which is the Fourier expansion of $I_h(\tau, f)$. (Since picking out the m-th Fourier coefficient is achieved by integrating over a circle, we can interchange the regularized integral with the 'Fourier integral'.) More precisely, let $f \in H_0^+(\Gamma)$ be a harmonic weak Maass form for Γ with constant terms $a_\ell^+(0)$ at the cusp ℓ. Then by Proposition 5.8 the m-th Fourier coefficient of the regularized lift is given by

$$\begin{aligned}\int_M^{\mathrm{reg}} f(z) \theta_{m,h}(\tau, z)\, d\mu(z) = \lim_{T \to \infty} \Bigg[&\int_{M_T} f(z) \theta_{m,h}(\tau, z)\, d\mu(z) \\ &- \frac{\log(T)}{\sqrt{N}} \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} a_\ell^+(0) \varepsilon_\ell b_\ell(m, h) \Bigg].\end{aligned} \tag{8.3}$$

Here M_T is the truncated surface M_T defined in (3.1) and $b_\ell(m,h)$ is the (m,h)-Fourier coefficient of the unary theta series $\tilde{\Theta}_{K_\ell}(\tau)$ as before. For $m \neq 0$, the set $\Gamma \backslash L_{m,h}$ is finite.

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Therefore, for these m, we see

$$\begin{aligned}(8.4)\qquad &\int_M^{\mathrm{reg}} f(z)\theta_{m,h}(\tau,z)\,d\mu(z)\\ &= \lim_{T\to\infty}\Big[\sum_{X\in\Gamma\backslash L_{m,h}} \int_{M_T} f(z) \sum_{\gamma\in\Gamma_X\backslash\Gamma} \varphi_0(X,\tau,\gamma z)\,d\mu(z)\\ &\qquad - \frac{\log(T)}{\sqrt{N}} \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} a_\ell^+(0)\varepsilon_\ell b_\ell(m,h)\Big].\end{aligned}$$

For non-zero m in the elliptic and the split hyperbolic situation we have seen in Section 7.2 that

$$\int_M \sum_{\gamma\in\Gamma_X\backslash\Gamma} f(z)\varphi_0(X,\tau,\gamma z)\,d\mu(z)$$

actually converges, corresponding to the fact that $b_\ell(m,h)=0$. Then the current equations in Section 7.2, Corollaries 7.5 and 7.7, give the Fourier coefficients for Theorem 4.1 for those m. In the next section we will consider the split hyperbolic periods, as well as the 0-th coefficient.

8.1. The split hyperbolic Fourier coefficients. We now consider the case m/N is a square, when the associated cycles are infinite geodesics. Throughout $X\in V$ denotes a vector of length $Q(X)=m$ with m/N is a square and $f\in H_0^+(\Gamma)$ is a harmonic weak Maass form.

In view of the characterization of the regularized integral in (8.4), we need to consider the behavior of $\int_{M_T}\sum_{\gamma\in\overline{\Gamma}} f(z)\varphi_0(X,\tau,\gamma z)\,d\mu(z)$ as $T\to\infty$.

Proposition 8.1. *The asymptotic behavior of*

$$\int_{M_T}\sum_{\gamma\in\overline{\Gamma}} f(z)\varphi_0(X,\tau,\gamma z)\,d\mu(z)$$

as $T\to\infty$ is given by

$$\begin{aligned}&\frac{1}{2}\Big(\int_{c_X}^{\mathrm{reg}} f(z)\,dz_X\Big)e(m\tau) - \frac{1}{2\sqrt{m}}\Big[\Big(\log 2\sqrt{\pi v m} + \frac{1}{2}\log 2 + \frac{1}{4}\gamma\\ &\qquad - \sqrt{\pi}\int_0^{2\sqrt{\pi v m}} e^{w^2}\operatorname{erfc}(w)\,dw\Big)(a_{\ell_X}^+(0)+a_{\ell_{-X}}^+(0))\\ &\quad + 2\pi\Big(\int_0^{\sqrt{v}} e^{4\pi m w^2}dw\Big)\Big(\sum_{n<0} a_{\ell_X}^+(n)e^{2\pi i\operatorname{Re}(c(X))n} + a_{\ell_{-X}}^+(n)e^{2\pi i\operatorname{Re}(c(-X))n}\Big)\\ &\quad + (a_{\ell_X}^+(0)+a_{\ell_{-X}}^+(0))\log(T)\Big]e(m\tau).\end{aligned}$$

Before we prove the proposition, we first show how this implies the formula for this Fourier coefficient given in Theorem 4.1. In view of (8.4) we only need to show

Lemma 8.2. *We have*

$$(8.5)\qquad \frac{1}{2\sqrt{m}}\sum_{X\in\Gamma\backslash L_{m,h}} (a_{\ell_X}^+(0)+a_{\ell_{-X}}^+(0)) = \frac{1}{\sqrt{N}}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} a_\ell^+(0)\varepsilon_\ell b_\ell(m,h).$$

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Proof. We can sort the infinite geodesics by the cusps ℓ to which they go. We define $\delta_\ell(m,h)$ to be 1 if there exists a $X \in L_{m,h}$ such that c_X ends at the cusp ℓ, that is, if X is perpendicular to ℓ. By [13, Lemma 3.7], there are either no or $2\sqrt{m/N}\varepsilon_\ell$ many X in $\Gamma\backslash L_{m,h}$ such that the corresponding c_X end in ℓ. Hence the left hand side of (8.5) is equal to

$$\frac{1}{\sqrt{N}} \sum_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} \varepsilon_\ell(\delta_\ell(m,h) + \delta_\ell(m,-h))a_\ell^+(0) = \frac{1}{\sqrt{N}} \sum_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} \varepsilon_\ell b_\ell(m,h) a_\ell^+(0).$$

This proves the lemma. □

The remainder of the section will be devoted to the proof of Proposition 8.1. We begin with a few lemmas.

Lemma 8.3. *Let $f \in H_0^+(\Gamma)$. Then*

$$\begin{aligned}\int_{M_T} f(z) \sum_{\gamma\in\overline{\Gamma}} \varphi_0(X,\tau,\gamma z)\, d\mu(z) = \frac{1}{2} e(m\tau) \int_{c_X^T} f(z)\, dz_X \\ + \frac{1}{2\pi i}\int_{\partial M_T} f(z) \sum_{\gamma\in\overline{\Gamma}} \partial\eta(X,\tau,\gamma z) \\ + \frac{1}{2\pi i}\int_{\partial M_T} \bar\partial f(z) \sum_{\gamma\in\overline{\Gamma}} \eta(X,\tau,\gamma z).\end{aligned}$$

Here $c_X^T = c_X \cap M_T$.

Proof. Proceed as in the proof of Proposition 7.6. Since in a truncated fundamental domain for Γ, the only singularities of $\sum_{\gamma\in\overline{\Gamma}} \partial\eta(X,\tau,\gamma z)$ are along c_X^T, everything goes through as before except that one obtains in addition the boundary terms above. □

Lemma 8.4. *For $f \in H_0^+(\Gamma)$, the differential $\bar\partial f(z)$ is rapidly decreasing and hence*

$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{\partial M_T} \bar\partial f(z) \sum_{\gamma\in\overline{\Gamma}} \eta(X,\tau,\gamma z) = 0.$$

The main task is to consider

$$\frac{1}{2\pi i}\int_{\partial M_T} f(z) \sum_{\gamma\in\overline{\Gamma}} \partial\eta(X,\tau,\gamma z). \tag{8.6}$$

By arguments exactly analogous to [7, Lemma 5.2] (where the integral of f against $\partial\xi(X)$ is considered), we see that the asymptotic behavior of (8.6) as $T\to\infty$ is the same as that of

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial M_{T,\ell_X}} f(z) \sum_{\gamma\in\overline{\Gamma}_{\ell_X}} \partial\eta(X,\tau,\gamma z) \\ + \frac{1}{2\pi i}\int_{\partial M_{T,\ell_{-X}}} f(z) \sum_{\gamma\in\overline{\Gamma}_{\ell_{-X}}} \partial\eta(X,\tau,\gamma z).\end{aligned} \tag{8.7}$$

Here $\partial M_{T,\ell_X}, \partial M_{T,\ell_{-X}}$ are the boundary components of M_T at the cusps $\ell_X, \tilde\ell_X = \ell_{-X}$ respectively. All other terms are rapidly decaying.

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The key is the asymptotic behavior of $\sum_{\gamma\in\overline{\Gamma}_{\ell_X}} \partial\eta(X,\tau,\gamma z)$, which is given in the next lemma.

Lemma 8.5. *Let $r \in \mathbb{Q}$ be the real part of the geodesic $c(X)$ and write $\alpha = \alpha_{\ell_X}$ for the width of the cusp ℓ_X. For a nonzero integer n define the function $g(n,y)$ by*

$$g(n,y) = \int_1^\infty e^{4\pi vm/w^2}\left(e^{-C(w)^2} - \frac{2\pi\sqrt{vm}}{w}\operatorname{erfc}(C(w))\right)\frac{dw}{w},$$

where

$$C(w) = \left(\frac{2\sqrt{\pi vm}}{w} + \frac{\pi nyw}{2\sqrt{\pi vm}\alpha}\right).$$

Then for $r < \operatorname{Re}(\sigma_{\ell_X} z) \leq r+\alpha$,

$$\begin{aligned}
&\frac{1}{2\pi i}\sum_{\gamma\in\overline{\Gamma}_{\ell_X}} \partial\eta(X,\tau,\gamma\sigma_{\ell_X} z)e(-m\tau)\\
&= \frac{1}{2\sqrt{m}\alpha}\Bigg[\sum_{n\neq 0} g(n,y)e\left(-\frac{n(z-r)}{\alpha}\right) - \sqrt{\pi}\int_0^{2\sqrt{\pi vm}} e^{w^2}\operatorname{erfc}(w)\,dw + \log\left(\frac{2\sqrt{\pi vm}}{y}\right)\\
&\qquad + \frac{1}{2}\psi\left(\frac{z-r}{\alpha}\right) + \frac{1}{2}\psi\left(1-\frac{z-r}{\alpha}\right) + \log\alpha + \frac{1}{2}\log(2) + \frac{1}{4}\gamma\Bigg]\,dz.
\end{aligned}$$

Proof. By applying σ_{ℓ_X} we can assume that

$$X = \sqrt{\frac{m}{N}}\begin{pmatrix} 1 & -2r\\ 0 & -1\end{pmatrix}$$

so that $c_X = \{z \in D : \operatorname{Re}(z) = r\}$ is a vertical geodesic. Hence ℓ_X represents the cusp ∞ and $\overline{\Gamma}_{\ell_X} = \left\{\left(\begin{smallmatrix} 1 & \alpha k\\ 0 & 1\end{smallmatrix}\right) : k\in\mathbb{Z}\right\}$ with $\alpha = \alpha_{\ell_X}$. Then (see also (7.5))

$$\frac{1}{2\pi i}\partial\eta(X,z) = -\sqrt{v}\frac{x-r}{y}\frac{1}{z-r}\left(\int_1^\infty e^{-4\pi vm\frac{(x+r)^2}{y^2}w^2}\,dw\right)e(m\tau)\,dz.$$

Replacing z by $z+r$, we can assume $r=0$. We set for $s\in\mathbb{C}$

$$\omega(z,s) = -\sqrt{v}\frac{x}{y}\frac{1}{z}\int_1^\infty e^{-4\pi vm\frac{x^2}{y^2}w^2} w^s\,dw,$$

so that

$$\frac{1}{2\pi i}\partial\eta(X,z) = \omega(z,0)e(m\tau)\,dz.$$

We also define

$$\Omega(z,s) := \sum_{n\in\mathbb{Z}} \omega(z+\alpha n, s),$$

so that

$$\frac{1}{2\pi i}\sum_{\gamma\in\overline{\Gamma}_{\ell_X}} \partial\eta(X,\tau,\gamma z) = \Omega(z,0)e(m\tau)\,dz.$$

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Note that $\Omega(z,s)$ is a holomorphic function in s. For $\mathrm{Re}(s) > -1$, we write

$$\Omega(z,s) = -\sqrt{v}\frac{1}{y}\sum_{n\in\mathbb{Z}}\frac{x+\alpha n}{z+\alpha n}\int_0^\infty e^{-4\pi v m\frac{(x+\alpha n)^2}{y^2}w^2} w^s\,dw \tag{8.8}$$

$$+\sqrt{v}\frac{1}{y}\sum_{n\in\mathbb{Z}}\frac{x+\alpha n}{z+\alpha n}\int_0^1 e^{-4\pi v m\frac{(x+\alpha n)^2}{y^2}w^2} w^s\,dw. \tag{8.9}$$

For the term on the right hand side of (8.8), we compute

$$-2^{-s-2}v^{-s/2}(\pi m)^{-(s+1)/2}\alpha^{-s-1}y^s\Gamma\left(\frac{s+1}{2}\right)\sum_{n\in\mathbb{Z}}\frac{\mathrm{sgn}(x+\alpha n)}{(\frac{z}{\alpha}+n)|\frac{x}{\alpha}+n|^s}. \tag{8.10}$$

Since $0 < x \le \alpha$, we see

$$\sum_{n\in\mathbb{Z}}\frac{\mathrm{sgn}(x+\alpha n)}{(\frac{z}{\alpha}+n)|\frac{x}{\alpha}+n|^s} = \sum_{n=0}^\infty\frac{1}{(\frac{z}{\alpha}+n)|\frac{x}{\alpha}+n|^s} + \sum_{n=0}^\infty\frac{1}{(n+(1-\frac{z}{\alpha}))|n+(1-\frac{x}{\alpha})|^s}.$$

Now (using (3.7))

$$\lim_{s\to 0+}\sum_{n=0}^\infty\frac{1}{(w+n)|w'+n|^s} - \frac{1}{(w'+n)^{s+1}} = \sum_{n=0}^\infty\frac{1}{w+n} - \frac{1}{w'+n} = -\psi(w)+\psi(w').$$

Since the constant term of the Laurent expansion of the Hurwitz zeta-function $H(w',s)$ at $s=1$ is $-\psi(w')$, we conclude

$$\sum_{n=0}^\infty\frac{1}{(w+n)|w'+n|^s} = \frac{1}{s} - \psi(w) + O(s).$$

Via (8.10) we therefore easily see

$$\begin{aligned}
&-\sqrt{v}\frac{1}{y}\sum_{n\in\mathbb{Z}}\frac{x+\alpha n}{z+\alpha n}\int_0^\infty e^{-4\pi v m\frac{(x+\alpha n)^2}{y^2}w^2} w^s\,dw \\
&= \frac{1}{2\sqrt{m}\alpha}\Bigg(-\frac{1}{s}+\frac{1}{2}\psi\left(\frac{z}{\alpha}\right)+\frac{1}{2}\psi\left(1-\frac{z}{\alpha}\right) \\
&\qquad -\frac{\Gamma'(\frac{1}{2})}{4\sqrt{\pi}} + \log\left(\frac{2\sqrt{\pi v m}\alpha}{y}\right)\Bigg) + O(s).
\end{aligned} \tag{8.11}$$

Note $\Gamma'(1/2) = -\sqrt{\pi}(2\log(2)+\gamma)$.

To compute (8.9), we first substitute $w \to \frac{1}{w}$ in the integral and obtain

$$\sqrt{v}\frac{1}{y}\sum_{n\in\mathbb{Z}}\frac{x+\alpha n}{z+\alpha n}\int_1^\infty e^{-4\pi v m\frac{(x+\alpha n)^2}{y^2w^2}} w^{-2-s}\,dw. \tag{8.12}$$

Now we apply Poisson summation. Using [7, Lemma 5.1], we see that (8.12) equals

$$\frac{1}{2\sqrt{m}\alpha}\sum_{n\in\mathbb{Z}} g(n,y,s)e\left(-\frac{nz}{\alpha}\right)$$

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with

$$g(n,y,s)=\int_1^\infty e^{4\pi vm/w^2}\Big(e^{-\left(\frac{2\sqrt{\pi vm}}{w}+\frac{\pi nyw}{2\sqrt{\pi vm}\alpha}\right)^2}\\ -\frac{2\pi\sqrt{vm}}{w}\operatorname{erfc}\Big(\frac{2\sqrt{\pi vm}}{w}+\frac{\pi nyw}{2\sqrt{\pi vm}\alpha}\Big)\Big)w^{-s}\frac{dw}{w}.$$

For $n\neq 0$, the function $g(n,z,s)$ is holomorphic at $s=0$, while for $n=0$ we have

$$\begin{aligned} g(0,z,s)&=\int_1^\infty w^{-s-1}\,dw-2\pi\sqrt{vm}\int_1^\infty e^{4\pi vm/w^2}\operatorname{erfc}\Big(\frac{2\sqrt{\pi vm}}{w}\Big)w^{-s-2}\,dw\\ &=\frac{1}{s}-\sqrt{\pi}\int_0^{2\sqrt{\pi vm}}e^{w^2}\operatorname{erfc}(w)\,dw+O(s).\end{aligned}$$

Combining this with (8.11) completes the proof of Lemma 8.5. □

Lemma 8.5 now immediately gives the next result.

Lemma 8.6. *Let $r\in\mathbb{Q}$ be the real part of the geodesic $c(X)$ and write $\alpha=\alpha_{\ell_X}$. Let*

$$f(\sigma_{\ell_X}z)=\sum_{n\in\mathbb{Z}}a^+_{\ell_X}(n)e^{2\pi inz/\alpha}+\sum_{n<0}a^-_{\ell_X}(n)e^{2\pi in\bar{z}/\alpha}$$

be the Fourier expansion of f at the cusp ℓ_X. Then

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial M_{T,\ell_X}}f(z)\sum_{\gamma\in\overline{\Gamma}_{\ell_X}}\partial\eta(X,\tau,\gamma z)\\ &=-\frac{1}{2\sqrt{m}}\sum_{n\neq 0}g(n,T)e\Big(\frac{nr}{\alpha}\Big)\Big(a^+_{\ell_X}(n)+a^-_{\ell_X}(n)e^{4\pi ny}\Big)\\ &\quad-\frac{1}{4\sqrt{m}\alpha}\int_{z=iT}^{iT+\alpha}\Big[\psi\Big(\frac{z}{\alpha}\Big)+\psi\Big(1-\frac{z}{\alpha}\Big)+2\log\alpha\Big]f(\sigma_{\ell_X}(z+r))\,dz\\ &\quad-\frac{1}{2\sqrt{m}}\Big(\frac12\log(2)+\frac14\gamma+\log\Big(\frac{2\sqrt{\pi vm}}{T}\Big)-\sqrt{\pi}\int_0^{2\sqrt{\pi vm}}e^{w^2}\operatorname{erfc}(w)\,dw\Big)a^+_{\ell_X}(0).\end{aligned}$$

We consider the first term on the right hand side in Lemma 8.6.

Lemma 8.7. *We have*

$$\begin{aligned}&-\frac{1}{2\sqrt{m}}\lim_{T\to\infty}\sum_{n\neq 0}g(n,T)e\Big(\frac{nr}{\alpha}\Big)\Big(a^+_{\ell_X}(n)+a^-_{\ell_X}(n)e^{4\pi ny}\Big)\\ &\qquad=2\pi\Big(\int_0^{\sqrt{v}}e^{4\pi mw^2}dw\Big)\Big(\sum_{n<0}a^+_{\ell_X}(n)e\Big(\frac{nr}{\alpha}\Big)\Big).\end{aligned}$$

Proof. The statement follows from $\operatorname{erfc}(t)=O(e^{-t^2})$ as $t\to\infty$ and $\operatorname{erfc}(t)=2$ as $t\to-\infty$. □

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Summarizing, in view of (8.7), using Lemmas 8.6, 8.7, and Theorem 3.3 we finally obtain the asymptotic behavior of (8.6).

Lemma 8.8. *The asymptotic behavior of*

$$\frac{1}{2\sqrt{m}}\int_{c_X^T} f(z)dz_X + \frac{1}{2\pi i}\int_{\partial M_T} f(z)\sum_{\gamma\in\bar{\Gamma}} \partial\eta(X,\tau,\gamma z)e(-m\tau)$$

as $T\to\infty$ is given by

$$\frac{1}{2}\int_{c_X}^{\mathrm{reg}} f(z)\,dz_X - \frac{1}{2\sqrt{m}}\biggl(\log\frac{2\sqrt{\pi v m}}{T} + \frac{1}{2}\log 2 + \frac{1}{4}\gamma - \sqrt{\pi}\int_0^{2\sqrt{\pi v m}} e^{w^2}\operatorname{erfc}(w)\,dw\biggr)\bigl(a_{\ell_X}^+(0) + a_{\ell_{-X}}^+(0)\bigr)$$
$$+ 2\pi\biggl(\int_0^{\sqrt{v}} e^{4\pi m w^2}\,dw\biggr)\biggl(\sum_{n<0} a_{\ell_X}^+(n)e\Bigl(\frac{n\operatorname{Re}(c(X))}{\alpha}\Bigr) + a_{\ell_{-X}}^+(n)e\Bigl(\frac{n\operatorname{Re}(c(-X))}{\alpha}\Bigr)\biggr).$$

Combining Lemma 8.8 with Lemma 8.4 and using Lemma 8.3 completes the proof of Proposition 8.1!

8.2. The parabolic Fourier coefficient. Let $f\in H_0^+(\Gamma)$. For $m=0$, note

$$\int_M^{\mathrm{reg}} f(z)\theta_{0,h}(\tau,z)\,d\mu(z) = \int_M^{\mathrm{reg}} f(z)\,d\mu(z) + \int_M^{\mathrm{reg}} \sum_{\substack{X\in L_{0,h}\\ X\neq 0}} f(z)\varphi_0(X,\tau,\gamma z)\,d\mu(z),$$

where

$$\int_M^{\mathrm{reg}} f(z)\,d\mu(z) = \lim_{T\to\infty}\int_{M_T} f(z)\,d\mu(z)$$

as in [7] and

$$\begin{aligned}(8.13)\qquad &\int_M^{\mathrm{reg}} f(z)\sum_{\substack{X\in L_{0,h}\\ X\neq 0}}\varphi_0(X,\tau,\gamma z)\,d\mu(z)\\ &= \lim_{T\to\infty}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)}\Biggl[\int_{M_T} f(z)\sum_{\gamma\in\Gamma/\Gamma_\ell}\sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}}\varphi_0(X,\tau,\gamma z)\,d\mu(z) - \frac{a_\ell^+(0)\varepsilon_\ell b_\ell(0,h)}{\sqrt{N}}\log T\Biggr]\end{aligned}$$

by (8.3). Note that $b_\ell(0,h)=1$ if and only if $\bar{h}=0$. Otherwise $b_\ell(0,h)=0$.

For $Q(X)=0$, the function $\eta(X,z)$ and its derivatives have no singularities in D so that the equation

$$dd^c\eta(X,z) = \varphi_0(X,z)$$

holds everywhere. The following lemma is therefore immediate.

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Lemma 8.9. *Let $f \in H_0^+(\Gamma)$. Fix a cusp ℓ. Then for T sufficiently large we have*

$$\begin{aligned}\int_{M_T} f(z) \sum_{\gamma\in\Gamma/\Gamma_\ell} \sum_{\substack{X\in\ell\cap L_{0,h}\\ x\neq 0}} \varphi_0(X,\tau,\gamma z)\, d\mu(z) &= \frac{1}{2\pi i}\Bigg[\int_{\partial M_T} f(z) \sum_{\gamma\in\Gamma/\Gamma_\ell} \sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}} \partial\eta(X,\tau,\gamma z) \\ &\quad + \int_{\partial M_T} \bar\partial f(z) \sum_{\gamma\in\Gamma/\Gamma_\ell} \sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}} \eta(X,\tau,\gamma z)\Bigg].\end{aligned}$$

As before, the second term on the right hand side in Lemma 8.9 vanishes in the limit. In view of (8.13) the following proposition gives the constant coefficient in Theorem 4.1.

Proposition 8.10. *Write $\ell \cap (L+h) = \mathbb{Z}\beta_\ell u_\ell + k_\ell u_\ell$ for some $0 \le k_\ell < \beta_\ell$. Then the asymptotic behavior as $T \to \infty$ of*

$$\frac{1}{2\pi i}\int_{\partial M_T} f(z) \sum_{\gamma\in\Gamma/\Gamma_\ell} \sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}} \partial\eta(X,\tau,\gamma z)$$

is given by

$$-a_\ell^+(0)\frac{\varepsilon_\ell}{2\sqrt{N}}\left[\log(4\beta_\ell^2\pi v) + \gamma + \psi\left(\frac{k_\ell}{\beta_\ell}\right) + \psi\left(1-\frac{k_\ell}{\beta_\ell}\right) - 2\log T\right].$$

Here we (formally) set $\psi(0) = -\gamma$, which is justified since $-\gamma$ is the constant term of the Laurent expansion of ψ at 0.

Proof. We have

$$\sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}} \partial\eta(X,\tau,z) = \sum_{n=-\infty}^{\infty}{}' \partial\eta(nu_\ell + h_\ell), z).$$

Here $\sum'$ indicates that we omit $n=0$ in the sum in the case of the trivial coset. We can assume that ℓ corresponds to the cusp ∞ so that $u_\ell = \left(\begin{smallmatrix}0 & \beta\\ 0 & 0\end{smallmatrix}\right)$ with $\beta = \beta_\ell$ and $h_\ell = \left(\begin{smallmatrix}0 & k\\ 0 & 0\end{smallmatrix}\right)$ for some $0 \le k = k_\ell < \beta$. We easily see

$$\frac{1}{2\pi i}\partial\eta(nX_\ell + h_\ell, z) = -\frac{\sqrt{v}}{2y}\left(\int_1^\infty e^{-\pi(n\beta+k)^2vNt/y^2}\frac{dt}{\sqrt{t}}\right)dz.$$

Hence

$$\sum_{\gamma\in\Gamma/\Gamma_\ell} \sum_{\substack{X\in\ell\cap L_{0,h}\\ X\neq 0}} \partial\eta(X,\tau,\gamma z)$$

is rapidly decaying at all cusps except ∞, and for that cusp in the limit all terms in the sum over Γ/Γ_ℓ vanish except $\gamma = 1$.

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We set

$$\Omega(s) = -\frac{\sqrt{v}}{2y} \sum_{n=-\infty}^{\infty}{}' \int_1^\infty e^{-\pi(n\beta+k)^2 vNt/y^2} t^s \frac{dt}{\sqrt{t}} \tag{8.14}$$

$$= -\frac{\sqrt{v}}{2y} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty e^{-\pi(n\beta+k)^2 vNt/y^2} t^s \frac{dt}{\sqrt{t}} \tag{8.15}$$
$$+ \frac{\sqrt{v}}{2y} \sum_{n=-\infty}^{\infty}{}' \int_0^1 e^{-\pi(n\beta+k)^2 vNt/y^2} t^s \frac{dt}{\sqrt{t}}.$$

For the first term in (8.15), we have

$$\begin{aligned} & -\frac{1}{2\beta\sqrt{\pi N}}\left(\frac{\pi\beta^2 vN}{y^2}\right)^{-s} \Gamma\left(s+\frac{1}{2}\right)\left(H\left(2s+1,\frac{k}{\beta}\right) + H\left(2s+1, 1-\frac{k}{\beta}\right)\right) \\ &= \frac{1}{2\beta\sqrt{N}}\left[\frac{-1}{s} + \log\left(\frac{\pi\beta^2 vN}{y^2}\right) + 2\log 2 + \gamma \right. \\ &\qquad \left. + \psi\left(\frac{k}{\beta}\right) + \psi\left(1-\frac{k}{\beta}\right)\right] + O(s). \end{aligned} \tag{8.16}$$

Here $H(s, w) = \sum_{n=0}^{\infty}(n+w)^{-s}$ denotes the Hurwitz zeta function, where for $w = 0$ we set $H(s, w) = \zeta(s)$. Then $H(s, w)$ has a simple pole at $s = 1$ with constant term $-\psi(w)$ in the Laurent expansion. With our convention for $\psi(0)$ above, (8.16) also holds for $k = 0$.

For the second term in (8.15), we substitute $t \to \frac{1}{t}$ and apply the theta transformation formula to obtain

$$\begin{aligned} & \frac{1}{2\beta\sqrt{N}} \int_1^\infty \sum_{n\in\mathbb{Z}} e^{2\pi i nk/\beta} e^{-\pi y^2 n^2 t/\beta^2 vN} t^{-s} \frac{dt}{t} - \delta_{k,0}\frac{\sqrt{v}}{y}\int_1^\infty t^{-s-3/2}\, dt \\ &= \frac{1}{2\beta\sqrt{N}}\frac{1}{s} + g(y) + O(s), \end{aligned} \tag{8.17}$$

for a function g with $\lim_{y\to\infty} g(y) = 0$. Combining (8.16) and (8.17) gives an expression for $\Omega(0)\, dz = \frac{1}{2\pi i}\partial\eta(nX_\ell + h_\ell, z)$ which we can easily integrate over $\partial M_{T,\ell}$ to obtain the lemma. □

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